Lecture Notes on  
Nonparametric Spectral Estimation  

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I. INTRODUCTION  

In this paper, we discuss the classical nonparametric methods for spectral estimation. In particular, we analyze the periodogram, correlogram, averaged periodogram and Blackman-Tukey spectral estimators. The content and notation follows mainly [1, Ch. 2] and in some parts [2]. A set of slides that presents a major part of the material can be found at [3, Lectures 2-3].  

The autocovariance function for a zero-mean (wide sense) stationary process \( y(t) \) is defined as  
\[ r(k) \triangleq \mathbb{E}\{y(t)y^*(t-k)\}. \]  

It is easy to see from the definition that the autocovariance function has the following property:  
\[ r(k) = r^*(-k). \]  

First definition of power spectral density (PSD):  
\[ \phi(\omega) \triangleq \sum_{k=-\infty}^{\infty} r(k)e^{-j\omega k}. \]  

Second definition of PSD:  
\[ \phi(\omega) \triangleq \lim_{N \to \infty} \mathbb{E}\left\{\frac{1}{N}\left|\sum_{t=1}^{N} y(t)e^{-j\omega t}\right|^2\right\}. \]  

Definitions (2) and (3) are equivalent under the assumption that the autocovariance decays sufficiently rapidly, so that [1]  
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{k=-N}^{N} |k| |r(k)| = 0. \]  

The problem of interest is to determine an estimate \( \hat{\phi}(\omega) \) of the power spectral density from a finite sequence of observations \( y(1), \ldots, y(N) \).  

II. PERIODOGRAM AND CORRELOGRAM METHODS  

The periodogram spectral estimator relies on the definition (3), and is computed as  
\[ \hat{\phi}_p(\omega) \triangleq \frac{1}{N} \left|\sum_{t=1}^{N} y(t)e^{-j\omega t}\right|^2. \]  

The correlogram spectral estimator relies on the definition (2), and is computed as  
\[ \hat{\phi}_c(\omega) \triangleq \sum_{k=-(N-1)}^{N-1} \hat{r}(k)e^{-j\omega k}, \]
where \( \hat{r}(k) \) denotes an estimate of the covariance \( r(k) \) obtained from the observations \( y(1), \ldots, y(N) \). There are two standard ways to obtain the estimated covariance:

\[
\hat{r}(k) = \frac{1}{N-k} \sum_{t=k+1}^{N} y(t)y^*(t-k), \quad 0 \leq k \leq N-1
\]

(7)

and

\[
\hat{r}(k) = \frac{1}{N} \sum_{t=k+1}^{N} y(t)y^*(t-k), \quad 0 \leq k \leq N-1.
\]

(8)

The autocovariance estimates for negative lags are constructed using (1), so that

\[
\hat{r}(-k) = \hat{r}^*(k), \quad 0 \leq k \leq N-1.
\]

The only difference between the two estimators (7) and (8) is the factor in front of the sum. Consider the expectation of the autocovariance estimate (8),

\[
\mathbb{E}\{\hat{r}(k)\} = \mathbb{E}\left\{ \frac{1}{N} \sum_{t=k+1}^{N} y(t)y^*(t-k) \right\} = \frac{1}{N} \sum_{t=k+1}^{N} \mathbb{E}\{y(t)y^*(t-k)\} = \frac{N-k}{N} r(k), \quad k \geq 0,
\]

(9)

\[
\mathbb{E}\{\hat{r}(-k)\} = \mathbb{E}\{\hat{r}^*(k)\} = \frac{N-k}{N} r^*(-k) = \frac{N-k}{N} r(-k), \quad k \geq 0,
\]

Hence, the autocovariance estimate (8) is biased. Similarly, it is easy to see that the autocovariance estimate (7) is unbiased for any \( N \). Even so, the biased estimator (8) is most commonly used for several reasons (cf. [1, pp. 24-25]). Hence, in the sequel we will consider the correlogram spectral estimator using the autocovariance estimate (8). It should be noted that the number of terms used to estimate the autocovariance decays with increasing \( k \). Therefore, the autocovariance estimates are very poor for large \( k \). For example, the estimates are computed from one term only at the extreme \( k = N-1 \).

**A. Equivalence of the Periodogram and the Correlogram**

In the following, we will show that the periodogram (5) and the correlogram (6) are in fact equivalent. Consider the signal

\[
x(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} y(k)e(t-k),
\]

where \( y(k) \) are fixed constants and \( e(t) \) is white noise with unit variance, i.e. \( \mathbb{E}\{e(t)e^*(s)\} = \delta_{t,s} \). This can be seen as a convolution, and therefore \( x(t) \) is the output of a filter with transfer function

\[
Y(\omega) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} y(k)e^{-j\omega k}.
\]

Since \( e(t) \) is white noise, its PSD is given by \( \phi_e(\omega) = 1 \) and

\[
\phi_x(\omega) = |Y(\omega)|^2 \phi_e(\omega) = \frac{1}{N} \left| \sum_{k=1}^{N} y(k)e^{-j\omega k} \right|^2 = \hat{\phi}_p(\omega).
\]

(10)
A straightforward computation of the autocovariance function (for $k \geq 0$) yields
\[ r_x(k) = \mathbb{E}\left\{ x(t)x^*(t - k) \right\} = \mathbb{E}\left\{ \frac{1}{\sqrt{N}} \sum_{p=1}^{N} y(p)e(t - p) \left( \frac{1}{\sqrt{N}} \sum_{s=1}^{N} y(s)e(t - k - s) \right)^* \right\} \]
\[ = \frac{1}{N} \sum_{p=1}^{N} \sum_{s=1}^{N} y(p)y^*(s) \mathbb{E}\left\{ e(t - p)e^*(t - k - s) \right\} \]
\[ = \frac{1}{N} \sum_{p=1}^{N} \sum_{s=1}^{N} y(p)y^*(s) \delta_{p,k+s} \]
\[ = \{ \text{substitute } p = s + k \} = \frac{1}{N} \sum_{p=k+1}^{N} y(p)y^*(p - k) \]
\[
= \begin{cases} \hat{r}(k), & k = 0, \ldots, N - 1 \\ 0, & k \geq N, \end{cases}
\]
where $\hat{r}(k)$ is given by (8). Now, we obtain the PSD from (2) as
\[ \phi_x(\omega) = \sum_{k=-\infty}^{\infty} r_x(k)e^{-j\omega k} = \sum_{k=-(N-1)}^{N-1} \hat{r}(k)e^{-j\omega k} = \hat{\phi}_c(\omega) \]  
(11)
Equations (10) and (11) show the equivalence of the periodogram and the correlogram methods.

### III. Properties of the Periodogram Method

In what follows, we will analyze some properties of the nonparametric spectral estimators.

#### A. Bias Analysis

Using the equivalence of the correlogram and the periodogram, we compute the expectation of the spectral estimate as
\[ \mathbb{E}\{ \hat{\phi}_p(\omega) \} = \mathbb{E}\{ \hat{\phi}_c(\omega) \} = \mathbb{E}\left\{ \sum_{k=-(N-1)}^{N-1} \hat{r}(k)e^{-j\omega k} \right\} = \sum_{k=-(N-1)}^{N-1} \mathbb{E}\{ \hat{r}(k) \} e^{-j\omega k}. \]

Using (9), we obtain
\[ \mathbb{E}\{ \hat{\phi}_p(\omega) \} = \sum_{k=-(N-1)}^{N-1} \left( 1 - \frac{|k|}{N} \right) r(k)e^{-j\omega k}. \]

Define the triangular, or Bartlett, window
\[ w_B(k) \triangleq \begin{cases} 1 - \frac{|k|}{N}, & 0 \leq |k| \leq N - 1, \\ 0, & \text{elsewhere} \end{cases} \]

Then, we can write the expectation as
\[ \mathbb{E}\{ \hat{\phi}_p(\omega) \} = \sum_{k=-\infty}^{\infty} w_B(k)r(k)e^{-j\omega k}. \]

The DFT of the product of two sequences is equal to the convolution of their respective DFTs. Hence,
\[ \mathbb{E}\{ \hat{\phi}_p(\omega) \} = \frac{1}{2\pi} \int_{\pi}^{\pi} \phi(\psi)W_B(\omega - \psi)d\psi, \]  
(12)
where
\[ W_B(\omega) = \frac{1}{N} \left( \frac{\sin (\omega N/2)}{\sin (\omega/2)} \right)^2 \]
is the DFT of the Bartlett window $w_B(k)$ shown in Figure 1.
Fig. 1. Weighting functions for the standard biased and unbiased covariance estimators respectively, for $N = 25$.

Note that, if instead we would have used the autocovariance estimate (7), that would correspond to a weighting function being a rectangular window,

$$w_R(k) \triangleq \begin{cases} 1, & 0 \leq |k| \leq N - 1, \\ 0, & \text{elsewhere}, \end{cases}$$

since $\mathbb{E} \{ \hat{r}(k) \} = r(k)$.

Consider the asymptotic expectation

$$\lim_{N \to \infty} \mathbb{E} \{ \hat{\phi}_p(\omega) \} = \lim_{N \to \infty} \sum_{k=-(N-1)}^{N-1} \left( 1 - \frac{|k|}{N} \right) r(k)e^{-j\omega k} = \phi_p(\omega) - \lim_{N \to \infty} \frac{1}{N} \sum_{k=-(N-1)}^{N-1} |k|r(k)e^{-j\omega k} = \phi_p(\omega),$$

where we used (4) in the final step. Hence, the periodogram is asymptotically unbiased.

B. Spectral Resolution

- The expectation of the estimator $\mathbb{E} \{ \hat{\phi}_p(\omega) \}$ can be interpreted as the output of a system with a weighting (or window) function $W_B(\omega)$ and the true PSD $\phi_p(\omega)$ as input.
• The periodogram is asymptotically unbiased (infinite length window). This corresponds to a window function which is a Dirac impulse. It is seen also from (12) that this would give the true spectrum.
• The window function should be as close as possible to a Dirac impulse. That is, we wish to have a narrow main lobe and a small number of sidelobes. The window approaches a Dirac impulse with increasing $N$.
• Leakage is caused by the sidelobes. Power from the frequency band that contains most of the power is transferred (leaks) to frequency bands that contain less power.
• Smearing is caused by the main lobe. The width of the main lobe causes a smoothing or smearing of the spectrum.
• Details in the spectrum that are separated by less than $1/N$ cannot be resolved. Hence, $1/N$ is called the spectral resolution limit. The half-power (3 dB) width of the main lobe of $W_B(\omega)$ is approximately $2\pi/N$ radians ($1/N$ in frequency). Figure 2 shows the 3 dB width of a Bartlett window for $N = 25$ and $N = 5$ respectively.

C. Variance Analysis

In what follows, we derive the (asymptotic) covariance of the periodogram spectral estimate for a zero-mean circularly symmetric complex white Gaussian noise process $\{y(t)\}$ (see Appendix A). For white
noise, \( \phi(\omega) = \sigma^2 \). Then, the covariance of the periodogram is computed as

\[
\text{Cov} \left\{ \hat{\phi}_p(\omega_1), \hat{\phi}_p(\omega_2) \right\} = \mathbb{E} \left\{ (\hat{\phi}_p(\omega_1) - \phi(\omega_1))(\hat{\phi}_p(\omega_2) - \phi(\omega_2)) \right\}
\]

\[
= \mathbb{E} \left\{ \hat{\phi}_p(\omega_1)\hat{\phi}_p(\omega_2) \right\} - \phi(\omega_1)\phi(\omega_2) = \mathbb{E} \left\{ \hat{\phi}_p(\omega_1)\hat{\phi}_p(\omega_2) \right\} - \sigma^4
\]

Using (5), we obtain

\[
\mathbb{E} \left\{ \left( \frac{1}{N} \sum_{t=1}^{N} y(t) e^{-j\omega_1 t} \right) \left( \frac{1}{N} \sum_{p=1}^{N} y(p) e^{-j\omega_2 p} \right) \right\} = \mathbb{E} \left\{ \frac{1}{N^2} \sum_{t=1}^{N} y(t) e^{-j\omega_1 t} \sum_{s=1}^{N} y(s) e^{-j\omega_1 s} \sum_{p=1}^{N} y(p) e^{-j\omega_2 p} \sum_{m=1}^{N} y^*(m) e^{-j\omega_2 m} \right\}
\]

\[
= \frac{1}{N^2} \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{s=1}^{N} \sum_{m=1}^{N} \mathbb{E} \left\{ y(t)y^*(s)y(p)y^*(m) \right\} e^{-j\omega_1 (t-s)} e^{-j\omega_2 (p-m)}
\]

To compute the expectation, we use the general result for the expectation of the product of four jointly Gaussian random variables. That is

\[
\mathbb{E} \{abcd\} = \mathbb{E} \{ab\} \mathbb{E} \{cd\} + \mathbb{E} \{ac\} \mathbb{E} \{bd\} + \mathbb{E} \{ad\} \mathbb{E} \{bc\} - 2 \mathbb{E} \{a\} \mathbb{E} \{b\} \mathbb{E} \{c\} \mathbb{E} \{d\}
\]

Then,

\[
\mathbb{E} \{y(t)y^*(s)y(p)y^*(m)\}
\]

\[
= \mathbb{E} \{y(t)y^*(s)\} \mathbb{E} \{y(p)y^*(m)\} + \mathbb{E} \{y(t)y(p)\} \mathbb{E} \{y^*(s)y^*(m)\} + \mathbb{E} \{y(t)y^*(s)\} \mathbb{E} \{y^*(p)y^*(m)\}
\]

\[
- 2 \mathbb{E} \{y(t)\} \mathbb{E} \{y^*(s)\} \mathbb{E} \{y(p)\} \mathbb{E} \{y^*(m)\}
\]

\[
= \sigma^4 (\delta_{t,s}\delta_{p,m} + \delta_{t,m}\delta_{p,s})
\]

Inserting (15) into (14) yields

\[
\mathbb{E} \left\{ \left( \frac{1}{N^2} \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{s=1}^{N} \sum_{m=1}^{N} \mathbb{E} \{y(t)y^*(s)y(p)y^*(m)\} e^{-j\omega_1 (t-s)} e^{-j\omega_2 (p-m)} \right) \right\}
\]

\[
= \frac{1}{N^2} \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{s=1}^{N} \sum_{m=1}^{N} \sigma^4 (\delta_{t,s}\delta_{p,m} + \delta_{t,m}\delta_{p,s}) e^{-j\omega_1 (t-s)} e^{-j\omega_2 (p-m)}
\]

\[
= \frac{1}{N^2} \sum_{t=1}^{N} \sum_{p=1}^{N} \sigma^4 e^{-j\omega_1 (t-s)} e^{-j\omega_2 (p-t)} = \sigma^4 + \frac{\sigma^4}{N^2} \sum_{t=1}^{N} e^{-j(\omega_1 - \omega_2)t} \sum_{s=1}^{N} e^{j(\omega_1 - \omega_2)s}
\]

Consider the sum of the second term, and note that it can be rewritten as

\[
\sum_{t=1}^{N} e^{-j(\omega_1 - \omega_2)t} \sum_{s=1}^{N} e^{j(\omega_1 - \omega_2)s} = \left| \sum_{t=1}^{N} e^{j(\omega_1 - \omega_2)t} \right|^2 = \left| \frac{e^{j(\omega_1 - \omega_2)N/2}}{e^{j(\omega_1 - \omega_2)/2}} - \frac{1}{1} \right|^2
\]

\[
= \left| \frac{e^{j(\omega_1 - \omega_2)N/2}}{e^{j(\omega_1 - \omega_2)/2}} - \frac{1}{1} \right|^2 \left( \sin ((\omega_1 - \omega_2)N/2) \right)^2
\]

If we now insert (17) and (16) into (13), we get

\[
\text{Cov} \left\{ \hat{\phi}_p(\omega_1), \hat{\phi}_p(\omega_2) \right\} = \sigma^4 + \frac{\sigma^4}{N^2} \left( \frac{\sin ((\omega_1 - \omega_2)N/2)}{\sin ((\omega_1 - \omega_2)/2)} \right)^2 - \sigma^4 = \frac{\sigma^4}{N^2} \left( \frac{\sin ((\omega_1 - \omega_2)N/2)}{\sin ((\omega_1 - \omega_2)/2)} \right)^2.
\]
Then, we get the asymptotic covariance by

$$\lim_{N \to \infty} \text{Cov} \left\{ \hat{\phi}_p(\omega_1), \hat{\phi}_p(\omega_2) \right\} = \lim_{N \to \infty} \frac{\sigma^4}{N^2} \left( \frac{\sin \left( \frac{(\omega_1 - \omega_2) N}{2} \right)}{\sin \left( \frac{(\omega_1 - \omega_2)}{2} \right)} \right)^2 = \begin{cases} \sigma^4, & \omega_1 = \omega_2, \\ 0, & \omega_1 \neq \omega_2. \end{cases}$$

The variance of the periodogram spectral estimator does not even asymptotically go to zero. That is, the periodogram is not a consistent estimator. The inconsistency holds also more generally, for colored Gaussian noise (cf. [1, Sec. 2.4.2]). Note further that spectral components at angular frequencies $\omega_1, \omega_2$ separated by $2\pi/N$ are uncorrelated. That is, the distance between neighboring uncorrelated samples decays with increasing $N$ and the spectral estimate fluctuates more rapidly.

**IV. AVERAGED PERIODOGRAM**

The main problem with the periodogram is the high variance (inconsistency). A simple solution to this is to average over a set of (ultimately independent) estimates (cf. [2, Sec. 4.4]). Suppose that $K$ independent data records of length $L$ are available, so that we can compute

$$\hat{\phi}_{p}^{(k)}(\omega) \triangleq \frac{1}{L} \left| \sum_{t=1}^{L} y(t) e^{-j\omega t} \right|^2, \quad k = 1, \ldots, K,$$

or equivalently $\hat{\phi}_{c}^{(k)}(\omega)$ from (6). Then, the averaged periodogram estimator is defined as

$$\hat{\phi}_w(\omega) \triangleq \frac{1}{K} \sum_{k=1}^{K} \hat{\phi}_{p}^{(k)}(\omega).$$

By the assumption that the $K$ data records are independent, we have

$$\mathbb{E} \left\{ \hat{\phi}_w(\omega) \right\} = \mathbb{E} \left\{ \hat{\phi}_p(\omega) \right\}$$

and

$$\text{Var} \left\{ \hat{\phi}_w(\omega) \right\} = \frac{1}{K} \text{Var} \left\{ \hat{\phi}_p(\omega) \right\},$$

where $\hat{\phi}_p(\omega)$ refers to the original periodogram (5) using $N$ samples. That is, the expectation remains the same, but the variance is decreased by a factor $K$. However, since each estimate $\hat{\phi}_{p}^{(k)}(\omega)$ is based on $L < N$ samples, the resolution ($1/L$) is worse. Hence, for maximum resolution $L$ should be chosen as large as possible. However, to reduce the variance we should choose $K = N/L$ large or equivalently $L$ small. Clearly there is a tradeoff between resolution and variance. In practice, we do not have $K$ independent data records, but rather the $N$ samples are segmented into $K$ (correlated) non-overlapping blocks of length $L$. Then, the variance reduction is in general greater than $1/K$.

**V. BLACKMAN-TUKEY SPECTRAL ESTIMATION**

In what follows, we briefly discuss the Blackman-Tukey method, which aims at reducing the statistical variability of the spectrum estimate, at the cost of spectral resolution (and bias). The high variance arises from the poor estimate of the autocorrelation for extreme lags, and the large number of small covariance errors that are summed. Moreover, we saw that the expectation of the periodogram can be seen as a windowing of the true spectrum. A windowing and a truncation of the sum (6) leads to the Blackman-Tukey estimator

$$\hat{\phi}_{\text{BT}}(\omega) = \sum_{k=-(M-1)}^{M-1} w(k) \hat{r}(k) e^{-j\omega k},$$

(19)
where \( M < N \) and \( w(k) \) is real sequence satisfying the following conditions
\[
0 \leq w(k) \leq w(0) = 1 \\
w(-k) = w(k) \\
w(k) = 0, \ |k| > M.
\]
The function \( w(k) \) weights the lags of the sample covariance, and is therefore called a *lag window*. In analogy with (12), we can rewrite (19) in the form
\[
\hat{\phi}_{\text{BT}}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\phi}_{p}(\psi) W(\omega - \psi) d\psi 
\]
Most windows in common use has a dominant relatively narrow peak (main lobe) around \( \omega = 0 \). Therefore, the Blackman-Tukey estimator (19) is a locally weighted average of the periodogram.

**A. Bias Analysis**

Since the periodogram is asymptotically unbiased, we have that
\[
E \left\{ \hat{\phi}_{\text{BT}}(\omega) \right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} E \left\{ \hat{\phi}_{p}(\psi) \right\} W(\omega - \psi) d\psi
\]
is asymptotically \((M, N \to \infty)\) unbiased. Moreover, the bias gets smaller with increasing \( M \), since then the spectral window approaches a Dirac impulse. In analogy with the periodogram, the Blackman-Tukey spectral estimator will (for most lag windows) be able to resolve details to about the bandwidth of the mainlobe of the spectral window, which in this case is approximately \( 1/M \). That is, the spectral resolution of the Blackman-Tukey estimator (19) is \( 1/M \), owing to the truncation of the sum.

**B. Variance Analysis**

It can be shown that the approximate variance for (19) is in the order of \( M/N \) for large data records (see [2, Appendix 4B] and [1]).

Clearly there is a variance/resolution (and a variance/bias) tradeoff, since the spectral resolution is \( O(1/M) \) and the variance is \( O(M/N) \). To get good spectral resolution (or a small bias) \( M \) should be chosen to be large, since then the spectral window will get closer to a Dirac impulse. On the other hand, for a small variance \( M \) should be chosen to be small.

**VI. Examples**

In what follows, we will show a few examples of the periodogram and the Blackman-Tukey spectral estimators. The examples are highly inspired by [2, Sec. 4.6]. Our example consist of the following sinusoidal signals in white noise
\[
y(t) = \sqrt{10} \exp(j2\pi 0.25t) + \sqrt{20} \exp(j2\pi 0.3t) + e(t), \quad (21)
\]
where \( e(t) \) is circularly symmetric complex Gaussian noise with unit variance.

Figure 3 shows the periodogram spectral estimates for \( N = 20 \). Figure 3(a) shows the estimates of 50 realizations in an overlaid format and Figure 3(b) shows the mean of the 50 realizations. Figure 4 shows the corresponding results for the Blackman-Tukey estimator, using a Bartlett window of length \( M = 5 \). It should be noted that the periodogram can resolve the two peaks, which are separated exactly at the resolution limit \( 1/N = 1/20 \). However, for the Blackman-Tukey estimator both peaks appear as one wide peak, since the spectral resolution is only \( 1/M = 1/5 \). On the other hand, it should also be noted that the estimate varies more for the periodogram shown in Figure 3(a) compared to the Blackman-Tukey estimate in Figure 4(a).

Figures 5 and 6 shows the corresponding results for when the data record length is increased to \( N = 100 \) and \( M = 20 \). Worth noting is that the B-T estimate is nearly able to resolve the two sinusoids since the resolution is now \( 1/M = 1/20 \). Moreover, the variance of the estimates does not decrease with an increase of the data record length.
Fig. 3. Periodogram spectral estimate of (21) for $N = 20$
Fig. 4. Blackman-Tukey spectral estimate of (21) for $N = 20$, using a Bartlett window of length $M = 5$.

(a) 50 overloaded realizations.

(b) Mean of 50 realizations.
Fig. 5. Periodogram spectral estimate of (21) for $N = 100$

(a) 50 overloaded realizations.

(b) Mean of 50 realizations.
Fig. 6. Blackman-Tukey spectral estimate of (21) for $N = 100$, using a Bartlett window of length $M = 20$. 

(a) 50 overloaded realizations.

(b) Mean of 50 realizations.
APPENDIX

A. Circularly Symmetric White Noise

A sequence \( \{e(t)\} \) is called complex circularly symmetric white noise if it satisfies

\[
E \{e(t) e^*(s)\} = \sigma^2 \delta_{t,s}, \\
E \{e(t) e(s)\} = 0.
\]

These requirements can be rewritten as

\[
\sigma^2 \delta_{t,s} = E \{e(t) e^*(s)\} = E \{(\text{Re}[e(t)] + j \text{Im}[e(t)])(\text{Re}[e(s)] - j \text{Im}[e(s)])\}
\]

\[
= E \{\text{Re}[e(t)]\text{Re}[e(s)] + \text{Im}[e(t)]\text{Im}[e(s)]\} + j E \{\text{Im}[e(t)]\text{Re}[e(s)] - \text{Re}[e(t)]\text{Im}[e(s)]\}
\]

\[
\iff \begin{cases}
E \{\text{Re}[e(t)]^2\} + E \{\text{Im}[e(t)]^2\} = \sigma^2, \\
E \{\text{Re}[e(t)]\text{Re}[e(s)]\} = -E \{\text{Im}[e(t)]\text{Im}[e(s)]\}, \ t \neq s \\
E \{\text{Im}[e(t)]\text{Re}[e(s)]\} = E \{\text{Re}[e(t)]\text{Im}[e(s)]\}, \ t \neq s
\end{cases}
\]

and

\[
0 = E \{e(t) e(s)\} = E \{(\text{Re}[e(t)] + j \text{Im}[e(t)])(\text{Re}[e(s)] + j \text{Im}[e(s)])\}
\]

\[
= E \{\text{Re}[e(t)]\text{Re}[e(s)] - \text{Im}[e(t)]\text{Im}[e(s)]\} + j E \{\text{Im}[e(t)]\text{Re}[e(s)] + \text{Re}[e(t)]\text{Im}[e(s)]\}
\]

\[
\iff \begin{cases}
E \{\text{Re}[e(t)]\text{Re}[e(s)]\} = E \{\text{Im}[e(t)]\text{Im}[e(s)]\}, \\
E \{\text{Im}[e(t)]\text{Re}[e(s)]\} = -E \{\text{Re}[e(t)]\text{Im}[e(s)]\}.
\end{cases}
\]

Combining these equations yields the equivalent properties

\[
E \{\text{Re}[e(t)]\text{Re}[e(s)]\} = \frac{\sigma^2}{2} \delta_{t,s}, \\
E \{\text{Im}[e(t)]\text{Im}[e(s)]\} = \frac{\sigma^2}{2} \delta_{t,s}, \\
E \{\text{Re}[e(t)]\text{Im}[e(s)]\} = 0.
\]

REFERENCES