Pilot Aided Channel Estimation for OFDM: a Separated Approach for Smoothing and Interpolation

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Abstract— **In this paper[1] pilot aided channel estimation (PACE) for OFDM is addressed. For PACE equidistantly spaced pilot symbols allow to reconstruct the channel response by means of interpolation. The optimum minimum mean squared error (MMSE) estimator performs smoothing and interpolation jointly. To reduce the complexity of the optimum MMSE estimator, we propose to separate the smoothing and interpolation tasks. The separated smoothing and interpolation estimator (SINE) consists of a MMSE based smoother which only operates at the received pilot symbols, and an interpolator which is independent of the channel statistics. We show that the separated approach gets close to the optimum MMSE, while the complexity is grossly reduced. However, at high SNR an error floor is observed, which is caused by edge effects, i.e. subcarriers near the beginning and end of the band suffer from an increased interpolation error.**

I. Introduction

Multi-carrier modulation, in particular orthogonal frequency division multiplexing (OFDM) [1], has emerged as an effective transmission technique for highly dispersive channels, and has been successfully applied to a wide variety of digital communications systems. In order to coherently detect a signal being transmitted over a multipath fading channel, accurate channel estimation is essential. For PACE, known pilot symbols are periodically inserted in the data stream. If the spacing of the pilots is sufficiently close to satisfy the sampling theorem, channel estimation and interpolation for the entire data sequence is possible [2]. PACE has been first applied to OFDM in [3].

The optimum solution for PACE is given by the Wiener interpolation filter (WIF). Unfortunately, an optimum WIF requires information about the channel statistics, which means that generating the filter coefficients is in many cases prohibitive. As an alternative, a WIF with model mismatch has been proposed [3, 4]. In this case the WIF is matched to a typical worst case scenario, so the filter coefficients can be precomputed and stored. However, for some scenarios, e.g. when the mismatch between actual and assumed channel statistics is large, the accuracy attained by the mismatched WIF may be insufficient.

In this paper, the objective is to close the performance gap between the matched and mismatched WIF. Unlike the WIF which jointly averages over the noise and interpolates, we follow a separated approach for smoothing and interpolation, as suggested in [5]. The motivation for the proposed smoothing and interpolation estimator (SINE) is twofold: first, according to the sampling theorem perfect interpolation of a noiseless signal is possible, without any knowledge of the channel statistics; second, a MMSE-based smoother which filters out noise at pilot positions only, can be implemented with significantly less computational cost. In fact, it is shown in the Appendix that the SINE approaches the performance of the optimum WIF; if the pilot sequence is of infinite length, and the interpolator is implemented by an ideal low-pass interpolation filter.

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In a case study, the effectiveness of the SINE is verified in terms of performance and computational complexity. The smoother is implemented by a low rank estimator, based on the singular value decomposition (SVD) [6]. For the interpolation filter an appropriately dimensioned mismatched WIF is used. For the considered OFDM system, the SINE performs close to the optimum MMSE. Only at high signal to noise ratios (SNR) an error floor is observed, which is due to subcarriers near the edges of the band.

While the complexity of the SINE is still significantly larger compared to a mismatched WIF, pre-smoothing could be optionally implemented, e.g. for high end terminals. In order to allow for a scalable and flexible receiver design the following estimator structure appears attractive: a compulsory mismatched WIF could be used for the interpolator, and an optional pre-smoothing may be applied for improved performance.

II. System & Channel Model

For OFDM the signal stream is divided into N_{c} parallel substreams. The n^{th} subcarrier of an OFDM symbol block is denoted by X_n. An inverse DFT with $N_{\mathrm{FFT}} \geq N_{\mathrm{c}}$ points is performed on each block, and subsequently a guard interval (GI) having N_{GI} samples is inserted, in the form of a cyclic prefix. Subsequently the signal is transmitted over a multipath fading channel. At the receiver the guard interval is removed and a DFT on the received block of signal samples is performed, to obtain the output of the OFDM demodulation Y_n. We assume the cyclic prefix to yield perfect orthogonality. Then, the received signal after OFDM demodulation is obtained

$$Y_n \;=\; X_n\, H_n + N_n \tag{1}$$

where X_n, H_n and N_n denote the transmitted symbol with energy per symbol of E_{s}, the channel transfer function (CTF), and the additive white Gaussian noise (AWGN) with zero mean and variance N_0, respectively.

The discussion in this paper is limited to channel estimation in frequency direction over one OFDM symbol block, i.e. variations of the received signal over time are not considered. However, for multi-carrier systems to operate in a mobile environment, the observed channel is typically correlated in two dimensions, frequency and time. The one dimensional channel estimation scheme described in this paper, can be extended to two dimensional channel estimator by using two cascaded one dimensional estimators [4].

Channel model: The channel transfer function (CTF), H_n, is obtained by sampling the analog CTF $H(f)$ at frequency instants $f = n/T$, where $T = N_{\text{FFT}} T_{\text{spl}}$ represents the OFDM symbol duration, and T_{spl} is the sample duration. The CTF, $H(f)$, is the Fourier transform of the channel impulse response (CIR), $h(t)$. Considering a frequency selective, Rayleigh fading channel, modeled by a tapped delay line with Q_0 non-zero taps [7], H_n can be described by

$$H_n = H(n/T) = \sum_{q=1}^{Q_0} h_q \, e^{-j2\pi\,\tau_q\, n/T} \qquad (2)$$

The channel of the q^{th} tap, h_q, impinging with time delay τ_q, is a wide sense stationary (WSS), complex Gaussian random variable with zero mean.

Correlation properties of the channel: We assume that all channel taps of the CIR are mutually uncorrelated. Then, the correlation function of the CTF in (2) between subcarriers μ and ν becomes

$$R_{HH}[\mu-\nu] \triangleq \mathrm{E}\{H_{n-\mu} H^*_{n-\nu}\} = \sum_{q=1}^{Q_0} \sigma_q^2 \, e^{-j2\pi\,\tau_q\cdot[\mu-\nu]/T} \quad (3)$$

where $\sigma_q^2 = \mathrm{E}\{|h_q|^2\}$ accounts for the average power of the q^{th} channel tap.

III. Pilot Aided Channel Estimation

PACE was first introduced for single carrier systems and required a flat-fading channel [2]. When applying PACE to multi-carrier systems, the pilots are periodically inserted in frequency direction. To this end, $N_{\text{p}} = \lceil N_{\text{c}}/D_{\text{f}} \rceil$ known pilot subcarriers are multiplexed into the N_{c} subcarriers, having an equidistant pilot spacing of D_{f} subcarriers, i.e. one pilot symbol is followed by $D_{\text{f}} - 1$ data symbols. To describe PACE it is useful to define a subset of the transmitted signal sequence containing only pilots, $\{\tilde{X}_{\tilde{n}}\} = \{X_n\}$, with $n = \tilde{n} D_{\text{f}}$ and $\tilde{n} = 0, \ldots, N_{\text{p}} - 1$.[2]

In vector-matrix notation the received pilot sequence of one OFDM symbol can be conveniently expressed as

$$\tilde{\mathbf{y}} \;=\; \tilde{\mathbf{X}}\,\tilde{\mathbf{h}} + \tilde{\mathbf{n}} \qquad (4)$$

where the transmitted pilot sequence, the CTF, and the AWGN term are given by

$$\begin{aligned} \tilde{\mathbf{X}} &= \mathrm{diag}\big(\tilde{X}_0, \cdots, \tilde{X}_{N_{\text{p}}-1}\big) \;\in\; \mathbb{C}^{N_{\text{p}} \times N_{\text{p}}} \\ \tilde{\mathbf{h}} &= \big[\tilde{H}_0, \cdots, \tilde{H}_{N_{\text{p}}-1}\big]^{\text{T}} \;\in\; \mathbb{C}^{N_{\text{p}} \times 1} \\ \tilde{\mathbf{n}} &= \big[\tilde{N}_0, \cdots, \tilde{N}_{N_{\text{p}}-1}\big]^{\text{T}} \;\in\; \mathbb{C}^{N_{\text{p}} \times 1} \end{aligned}$$

[2]As a general convention, variables referring to the positions of the pilot symbols will be marked with a ˜ in the following.

The first step in the channel estimation process is to remove the modulation of the pilot symbols. Thus, an initial estimate of the CTF at pilot positions is obtained

$$\breve{\mathbf{h}} \;=\; \tilde{\mathbf{X}}^{-1}\tilde{\mathbf{y}} \;=\; \tilde{\mathbf{h}} + \tilde{\mathbf{X}}^{-1}\tilde{\mathbf{n}} \qquad (5)$$

which corresponds to least squares (LS) estimate.

A. Sampling Theorem and Edge Effects

Given that pilots are inserted with rate D_{f}, the CTF at pilot positions is sampled with rate D_{f}/T. This results in periodic replicas of the spectrum of $H(nD_{\text{f}}/T)$ with distance T/D_{f}, known as aliases. In order to prevent overlapping of the original spectrum with its aliases, there exists a maximum D_{f}, dependent on the maximum delay of the channel, τ_{max}. The sampling theorem requires that $\tau_{max} \leq T/D_{\text{f}}$ [8].

When applying the sampling theorem to discrete waveforms it is inherently assumed that the signal has infinite duration. Particularly, near the first and last subcarriers, this assumption is violated and edge effects are observed, resulting in an unavoidable increase of the estimation error.

B. Channel Estimation by Wiener Filtering

The Wiener interpolation filter (WIF) is implemented by an FIR filter with M_{wf} taps. Generally, it is of great computational complexity to use all available pilots. Instead a window of size M_{wf} can be slid over the frequency grid, with $M_{\text{wf}} < N_{\text{p}}$. If possible the desired symbol should be placed in the center of the sliding widow. However, near the band edges, i.e. the beginning and end of the OFDM symbol block, this is not possible. Mathematically speaking, the subset of demodulated pilots within the sliding window is denoted by

$$\breve{\mathbf{h}}_n = \begin{cases} \big[\breve{H}_0, \cdots, \breve{H}_{M_{\text{wf}}-1}\big]^{\text{T}}; & n \leq \lfloor \frac{M_{\text{wf}}}{2} \rfloor D_{\text{f}} \\ \big[\breve{H}_{N_{\text{p}}-M_{\text{wf}}}, \cdots, \breve{H}_{N_{\text{p}}-1}\big]^{\text{T}}; & n \geq N_{\text{c}} - \lfloor \frac{M_{\text{wf}}}{2} \rfloor D_{\text{f}} \\ \big[\breve{H}_{\lfloor \frac{n}{D_{\text{f}}} \rfloor - \lfloor \frac{M_{\text{wf}}}{2} \rfloor - 1}, \cdots, \breve{H}_{\lfloor \frac{n}{D_{\text{f}}} \rfloor + \lfloor \frac{M_{\text{wf}}}{2} \rfloor}\big]^{\text{T}}; & \text{otherwise} \end{cases} \qquad (6)$$

The 1^{st}, 2^{nd} and 3^{rd} entry of $\breve{\mathbf{h}}_n$ correspond to the sliding window placed at the beginning, end and center of the OFDM symbol block, respectively Then, the channel estimate for subcarrier n is determined by

$$\widehat{H}_n = \mathbf{w}^{\text{H}}[n] \cdot \breve{\mathbf{h}}_n \qquad (7)$$

Not only the quality of $\widehat{H}_n$, also the filter $\mathbf{w}[n] = [W_1[n], \cdots, W_{M_{\text{wf}}}[n]]^{\text{T}}$ depends on the location of the desired symbol within the OFDM symbol block. However, within the center region of the OFDM symbol, $\mathbf{w}[n]$ is periodic with D_{f}, so $\mathbf{w}[n] = \mathbf{w}[n \bmod D_{\text{f}}]$. This means that in total $(M_{\text{wf}} - 1)D_{\text{f}} + 1$ different filters are required to estimate the entire OFDM symbol block.

The optimum WIF which minimizes the mean squared error (MSE) is obtained by solving the Wiener-Hopf equation, that is

$$\mathbf{w}_{\text{o}}[n] = \mathbf{R}^{-1}_{\breve{\mathbf{h}}\breve{\mathbf{h}}}\, \mathbf{r}_{\breve{\mathbf{h}}H}[n] \quad \in \mathbb{C}^{M_{\text{wf}} \times 1} \qquad (8)$$

where the auto-correlation matrix and the cross-correlation vector are given by [9]

$$\mathbf{R}_{\breve{\mathbf{h}}\breve{\mathbf{h}}} = \mathrm{E}\{\breve{\mathbf{h}}_n\, \breve{\mathbf{h}}_n^{\text{H}}\} = \mathbf{R}_{\tilde{\mathbf{h}}\tilde{\mathbf{h}}} + \frac{N_0}{E_s}\,\mathbf{I} \quad \in \mathbb{C}^{M_{\text{wf}} \times M_{\text{wf}}}$$

$$\mathbf{r}_{\breve{\mathbf{h}}H}[n] = \mathrm{E}\{\breve{\mathbf{h}}_n H_n^*\} = \mathbf{r}_{\tilde{\mathbf{h}}H}[n] \quad \in \mathbb{C}^{M_{\text{wf}}\times 1} \tag{9}$$

The μ^{th} column and ν^{th} row of $\mathbf{R}_{\breve{\mathbf{h}}\breve{\mathbf{h}}}$ is given by $R_{HH}[(\mu-\nu)D_{\text{f}}]$ from (3). The μ^{th} entry of $\mathbf{r}_{\breve{\mathbf{h}}H}[n]$ accounts for the correlation between the μ^{th} pilot within the sliding window and the desired symbol. Hence, the cross correlation term becomes $R_{HH}[\mu D_{\text{f}} - \Delta n]$, with

$$\Delta n = \begin{cases} n & n \leq \lfloor \frac{M_{\text{wf}}}{2} \rfloor D_{\text{f}} \\ n - N_{\text{c}} + (M_{\text{wf}}-1)D_{\text{f}}+1 & n \geq N_{\text{c}} - \lfloor \frac{M_{\text{wf}}}{2} \rfloor D_{\text{f}} \\ (n \mod D_{\text{f}}) - \lfloor \frac{M_{\text{wf}}-1}{2} \rfloor D_{\text{f}} & \text{otherwise} \end{cases} \tag{10}$$

If all available pilots are used, the WIF achieves the MMSE. Hence, the WIF of dimension $M_{\text{wf}} = N_{\text{p}}$, which is matched to the channel statistics will be referred to as MMSE estimator in the following.

1) Mismatched WIF: For the WIF the auto and cross-correlation functions need to be estimated at the receiver. More importantly, a computational costly matrix inversion in (8) may be in many cases prohibitive. Alternatively, a robust estimator with model mismatch may be chosen [4]. That is to assume a uniform power delay profile with maximum delay, T_{w}, which is to be expected in a certain transmission scenario, i.e. worst case propagation delay. The Fourier transform of a uniform power delay profile which is non-zero within the range $[0, T_{\text{w}}]$, yields the frequency correlation between subcarriers μ and ν

$$R'_{HH}[\mu-\nu] = \frac{T \sin\left(\pi T_{\text{w}} \cdot [\mu-\nu]/T\right)}{\pi T_{\text{w}} \cdot [\mu-\nu]} \cdot e^{-j\pi T_{\text{w}} \cdot [\mu-\nu]/T} \tag{11}$$

In a well designed OFDM system non-zero channel taps should only occur within the guard interval $[-T_{\text{GI}}, 0]$. Thus, T_{w} is set equal to T_{GI} in the following.

By replacing the true correlation function in (8) with $R'_{HH}[\mu-\nu]$, the mismatched WIF is determined by

$$\mathbf{w}'[n] = \left(\mathbf{R}'_{\tilde{\mathbf{h}}\tilde{\mathbf{h}}} + \tfrac{1}{\gamma_{\text{w}}} \mathbf{I}\right)^{-1} \mathbf{r}'_{\tilde{\mathbf{h}}H}[n] \quad \in \mathbb{C}^{M_{\text{itp}}\times 1} \tag{12}$$

where γ_{w} denotes the average SNR at the filter input, which is used to generate the filter coefficients. Note γ_{w} should be equal or larger than actual average SNR, so $\gamma_{\text{w}} \geq \gamma_{\text{c}}$. Hence, in order to determine the channel estimator only T_{w} and the highest expected SNR γ_{w} are required. By using a mismatched estimator the filter coefficients can be precomputed and stored.

C. Separating Smoothing and Interpolation

The channel estimator should work for a wide range of different operation scenarios, e.g. outdoor or indoor environment. Moreover, in many cases the nature of a multipath fading channel is such that the number of non-zero channel taps is much smaller than the maximum delay of the channel in samples. This implies that if a WIF with model mismatch is used, there may be a severe mismatch between assumed and actual channel statistics. While for some applications this performance degradation of the mismatched WIF is acceptable, this may generally not be the case. For instance, a pilot boost may be avoided by applying more sophisticated channel estimation schemes.

Fig. 1 illustrates the general structure of the separated smoother and interpolation estimator (SINE). The smoother, $\mathbf{S}$,

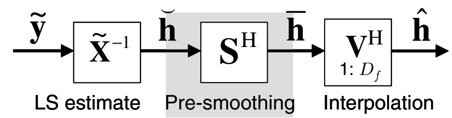

Fig. 1. Separated smoothing and interpolation estimator (SINE).

is employed to compensate for the noise at pilot positions only, by minimizing the MSE. Subsequently, the smoothed channel estimates at pilot positions are fed to an interpolation filter, $\mathbf{V}$, which is independent of the channel statistics, to yield channel estimates for all subcarriers.

The smoother of dimension $\mathbb{C}^{M_{\text{sm}}\times M_{\text{sm}}}$ is implemented by the filter bank $\mathbf{S} = [\mathbf{s}_1, \cdots, \mathbf{s}_{M_{\text{sm}}}]$, and produces the output $\bar{\mathbf{h}}_n = \mathbf{S}^{\text{H}}\breve{\mathbf{h}}_n$ at pilot positions. If $\mathbf{S}$ is determined by the MMSE criterion, we obtain

$$\mathbf{S}_{\text{o}} = \left(\mathbf{R}_{\tilde{\mathbf{h}}\tilde{\mathbf{h}}} + \tfrac{N_0}{E_s}\mathbf{I}\right)^{-1} \mathbf{R}_{\tilde{\mathbf{h}}\tilde{\mathbf{h}}} \quad \in \mathbb{C}^{M_{\text{sm}}\times M_{\text{sm}}} \tag{13}$$

Note, the m^{th} column of $\mathbf{S}_{\text{o}}$ is given by the WIF $\mathbf{w}_{\text{o}}[n]$ from (8), with Δn being a multiple of D_{f}. If the smoother uses all available pilots, the estimator dimension becomes $M_{\text{sm}} = N_{\text{p}}$.

The final channel estimate can be described by the concatenation of the smoother, $\mathbf{S}$, and interpolator, $\mathbf{v}[n]$, given by

$$\widehat{H}_n = \mathbf{v}^{\text{H}}[n] \cdot \mathbf{S}^{\text{H}}\breve{\mathbf{h}}_n = \mathbf{v}^{\text{H}}[n] \cdot \bar{\mathbf{h}}_n \tag{14}$$

Suitable smoothing and interpolation filters for the SINE are discussed in sections IV-A and IV-B.

The SINE has the following properties:

- It is shown in Appendix A that the SINE approaches the MMSE, if $\mathbf{S}_{\text{o}}$ is generated according to (13), and $\mathbf{v}[n]$ is an ideal lowpass interpolation filter; which per definition is independent of the channel statistics.
- In the results section IV-D we demonstrate that the performance gets close to the MMSE, even for realizable dimensions of $\mathbf{v}[n]$. However, edge effects cause an error floor at high SNR.
- The design of a purely pilot aided adaptive estimator is possible, as there are no cross-correlation terms between data and pilot subcarriers. Hence, decision feedback effects, caused by erroneous decisions on data symbols can be avoided. This may prove particularly useful when high modulation cardinalities are employed.
- The observation space for algorithms which track the channel statistics is reduced D_{f} times. Thus, the complexity of adaptive estimators, e.g. a Kalman filter [9] can be significantly reduced.
- The SINE allows for a scalable receiver design, in the way that pre-smoothing may only be applied optionally, in case the performance of a mismatched WIF is insufficient.

IV. Estimator Design and Performance

In this section a case study for the implementation of the SINE is described. For pre-smoothing an adaptive Kalman filter [10, 11] may be employed. In this paper, however, we follow a different approach approach, low rank estimation based on the singular value decomposition (SVD) [6].

Unlike the smoother, the interpolator $\mathbf{v}[\Delta n]$ is designed independent of the channel statistics. Interpolation may be performed by low order polynomial interpolation [5], or by a low-pass interpolation filter with windowing [12]. However, we prefer a mismatched WIF, since superior performance is expected near the band edges, compared to low-pass interpolation. Furthermore, a 1st or 2nd order polynomial interpolator as proposed in [5], exhibits a larger interpolation error in dispersive channels.

A. Smoothing: Low Rank Estimator (LRE)

For pre-smoothing an adaptive Kalman filter [10, 11], or a low rank estimator (LRE) based on the singular value decomposition (SVD) may be employed [6]. In this paper we follow the latter approach, SVD based pre-smoothing.

For low rank approximation based on the SVD, the received pilot sequence is transformed, such that all of its components become mutually uncorrelated [13]. The motivation to use a low rank estimator is that the channel is generally of sparse nature. The rank, which corresponds to the number of uncorrelated fading taps of the channel, is usually much smaller than the maximum delay of the channel, so $Q_0 \ll \tau_{max}/T_{\rm spl} \leq N_{\rm GI}$.

Although it is possible to find a general low rank approximation for PACE, which jointly performs smoothing and interpolation, it is not very practical. The estimator can be grossly simplified if the LRE is used for smoothing only [5]. Given the SVD[3] of the auto-correlation matrix $\mathbf{R}_{\tilde{\mathbf{h}}\tilde{\mathbf{h}}} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\rm H}$, the channel estimate of one OFDM symbol containing $M_{\rm sm}$ subcarriers can be expressed as [6]

$$\bar{\mathbf{h}} = \mathbf{U}\mathbf{D}\mathbf{U}^{\rm H}\,\breve{\mathbf{h}} \approx \mathbf{U}\mathbf{D}_\rho\mathbf{U}^{\rm H}\,\breve{\mathbf{h}} \tag{15}$$

The smoother matrix from (13) is given by $\mathbf{S}_{\rm o} = \mathbf{U}\mathbf{D}\mathbf{U}^{\rm H}$. The unitary matrix $\mathbf{U} \in \mathbb{C}^{M_{\rm sm}\times M_{\rm sm}}$ contains the singular vectors of $\mathbf{R}_{\tilde{\mathbf{h}}\tilde{\mathbf{h}}}$. The real valued diagonal matrix $\mathbf{D} \in \mathbb{R}^{M_{\rm sm}\times M_{\rm sm}}$ has the entries

$$d_m = \frac{\lambda_m}{\lambda_m + \frac{N_0}{E_s}}\ , \quad m = \{1, \cdots, M_{\rm sm}\} \tag{16}$$

where λ_m is the $m^{\rm th}$ singular value, associated to the $m^{\rm th}$ column of $\mathbf{U} \in \mathbb{C}^{M_{\rm sm}\times M_{\rm sm}}$. The singular values are sorted in descending order such that λ_m is the mth largest entry. The optimum rank-ρ approximation is obtained by selecting the ρ largest entries of $\mathbf{D}$, $\{d_1, \cdots, d_\rho\}$, and setting the other entries to zero, to yield $\mathbf{D}_\rho = {\rm diag}[d_1, \cdots, d_\rho, 0, \cdots, 0]$. If $\rho = M_{\rm sm}$, the estimator approaches the MMSE.

B. Interpolation: Mismatched WIF

The mismatched WIF is denoted by $\mathbf{v}'[\Delta n]$, and implemented according to (12). According to the discussion in section III-B.1, the constant $\gamma_{\rm w}$ should be equal or larger than the highest SNR expected at the output of the smoother. In fact, $1/\gamma_{\rm w} = \varepsilon$ should be an as small as possible constant; but large enough to maintain numerical stability for the matrix inversion in (12).

Provided the dimension of $\mathbf{v}'[n]$ approaches infinity and $\varepsilon \to 0$, the interpolator, $\mathbf{v}'[n]$, becomes an ideal low-pass interpolation filter. According to Appendix A, the SINE will asymptotically approach the optimum MMSE estimator.

[3]For the considered problem the SVD becomes an Eigenvalue decomposition (EVD). However, in order to be consistent with the terminology in the literature, the term SVD will be used nevertheless

TABLE I

POWER DELAY PROFILE OF THE CHOSEN CHANNEL MODEL

Delay [ns]	0	10	30	360	370	385
Power [dB]	−3.00	−5.22	−6.98	−5.22	−7.44	−9.20
Delay [ns]	250	260	280	1040	1045	1065
Power [dB]	−4.72	−6.94	−8.70	−8.19	−10.41	−12.17
Delay [ns]	2730	2740	2760	4600	4610	4625
Power [dB]	−12.05	−14.27	−16.03	−15.50	−17.72	−19.48

C. Computational Complexity

In order to quantify the complexity of different channel estimation schemes we distinguish between *offline* and *online* complexity.

Offline complexity accounts for the generation of the filter coefficients. Since the channel statistics change relatively slowly, the filter coefficients need only to be updated in the order of several ms or so. While for the WIF matched to the channel statistic a matrix inversion in the order of $\mathcal{O}(M^3)$ is required, the SINE has only an offline complexity of $\mathcal{O}(M^2)$, where M is the filter order.

The online complexity represents the number of (in general complex valued) multiplications per subcarrier. The SINE, which is implemented by two cascaded filters, has an online complexity of

$$C = M_{\rm itp} + M_{\rm sm}/D_{\rm f} \tag{17}$$

Note, that the complexity of the smoother, $M_{\rm sm}$, is divided by $D_{\rm f}$. This is of particular advantage for large pilot spacings $D_{\rm f}$.

The online complexity can be further reduced by applying the low rank estimator described in section IV-A, to $C = M_{\rm itp} + 2\rho/D_{\rm f}$ multiplications per tone [6]. In comparison the WIF requires $M_{\rm wf}$ multiplications per tone.

D. Performance Evaluation

An OFDM system with $N_{\rm c} = 1024$ subcarriers, and a guard interval (GI) duration of $N_{\rm GI} = 100 \cdot T_{\rm spl}$ is used to evaluate the performance of the SINE. The signal bandwidth was set to 20 MHz, which corresponds to a sampling duration of $T_{\rm spl} = 50$ ns. The channel is modeled by a tap delay line model with a power delay profile as shown in Table I.

The MSE of an arbitrary estimator $\mathbf{w}$ of dimension $M_{\rm w} \times 1$ can be expressed in the general form [4]

$$\begin{aligned}{\rm MSE} &\triangleq {\rm E}\big[|H_n - \widehat{H}_n|^2\big] \\ &= {\rm E}\big[|H_n|^2\big] - 2\,{\rm Re}\{\mathbf{w}^{\rm H}\mathbf{r}_{\breve{\mathbf{h}}H}\} + \mathbf{w}^{\rm H}\mathbf{R}_{\breve{\mathbf{h}}\breve{\mathbf{h}}}\mathbf{w}\end{aligned} \tag{18}$$

with $\widehat{H}_n = \mathbf{w}^{\rm H}\breve{\mathbf{h}}_n$. The pilot spacing must satisfy the sampling theorem, which requires that $D_{\rm f} \leq T/\tau_{max}$ [8]. Since τ_{max} is not known we upper bound τ_{max} by the guard interval duration, so $D_{\rm f} \leq N_{\rm FFT}/N_{\rm GI}$. With the given parameters we obtain $D_{\rm f} \leq 10$. In the following we choose $D_{\rm f} = 8$, corresponding to a pilot oversampling factor of $\beta_{\rm f} = \frac{T}{T_{\rm w}D_{\rm f}} \approx 20\%$.

In Figures 2 and 3 a pre-smoother using all available pilots was used, so $M_{\rm sm} = N_{\rm p} = 128$. The rank, ρ of the SVD based smoother, described in section IV-A, was set sufficiently large to capture all significant singular values. The interpolation part is implemented by a mismatched WIF of dimension $M_{\rm itp}$.

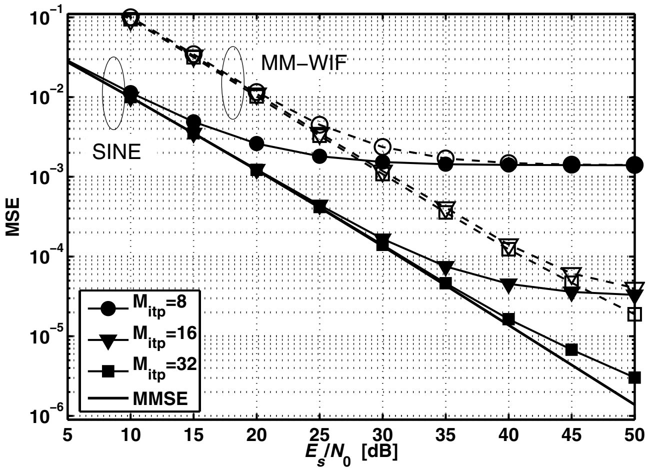

Fig. 2. MSE vs SNR for the SINE and the WIF with model mismatch.

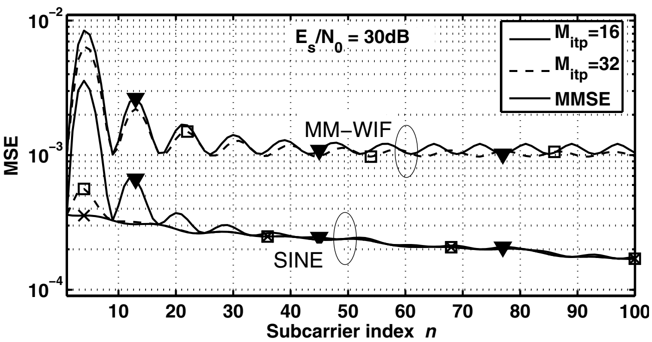

Fig. 3. MSE vs subcarrier index n, for the SINE in comparison to the WIF with model mismatch. $E_\mathrm{s}/N_0 = 30\,\mathrm{dB}$.

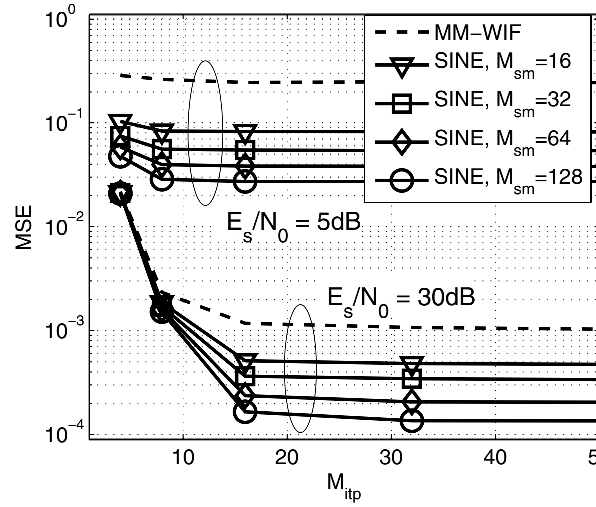

Fig. 4. MSE vs interpolator order M_itp, for the SINE having a pre-smoother with M_sm coefficients. $E_\mathrm{s}/N_0 = 5$ and $30\,\mathrm{dB}$.

Fig. 2 shows the MSE averaged over all subcarriers against the SNR, drawn with filled markers. For comparison, results for the mismatched WIF of dimension $M_\mathrm{wf} = M_\mathrm{itp}$ without pre-smoothing (transparent markers), and the MMSE as a lower bound, are also included.[4] It is seen that the MSE of the SINE approaches the MMSE at low SNR. At high SNR the interpolation error becomes dominant. By increasing M_itp the error floor can be lowered almost arbitrarily. However, at very high SNR, the performance of the SINE and mismatched WIF will merge, for arbitrary M_itp.

To illustrate edge effects, in Fig. 3 the MSE performance is plotted against the subcarrier index n of the first 100 subcarriers, at an SNR of $E_\mathrm{s}/N_0 = 30\,\mathrm{dB}$. Near the band edge, ripples in between pilot positions are observed, due to an irreducible interpolation error. Unlike for the mismatched WIF, increasing the interpolator dimension of the SINE from $M_\mathrm{itp} = 16$ to 32 coefficients, significantly reduces these ripples. Further towards the center of the band ($n \geq 30$), for the SINE ripples completely disppear, and the performance very closely matches the MMSE. In any case, the deviation to the overall MMSE averaged over all subcarriers is mainly due to edge effects, which corrupt only a few % of all subcarriers.

[4]The MMSE is attained by a perfectly matched WIF using all available pilots, so $M_\mathrm{wf} = N_\mathrm{p} = 128$.

Since the computational cost is mainly determined by the dimension of the estimator, it is instructive to examine the performance with respect to the estimator dimensions of the smoother and interpolator. In Fig. 4 the MSE is plotted against the interpolator dimension M_itp, i.e. the dimension of the mismatched WIF, for various smoother dimensions M_sm. It is seen that the MSE decreases inversely proportional to the smoother dimension M_sm. For the interpolator, the MSE saturates for $M_\mathrm{itp} \geq 8$ ($M_\mathrm{itp} \geq 16$), at an SNR of $E_\mathrm{s}/N_0 = 5\,\mathrm{dB}$ ($30\,\mathrm{dB}$), independent of M_sm. Thus, unlike for the smoother dimension M_sm, nothing is gained by exceeding M_itp beyond 16.

V. Conclusions

In this paper a separated approach for smoothing and interpolation was presented. The proposed smoothing and interpolation estimator (SINE) consists of a smoother, which filters out noise at the received pilot subcarriers only. The subsequent interpolation part can be realized without any knowledge of the channel statistics. For the considered case study, performance close the optimum MMSE can be achieved, even for an SNR up to 30 dB. Moreover, the complexity with respect to the optimum MMSE estimator is significantly reduced. Only at very high SNR, edge effects can cause an irreducible error floor.

A smoother and interpolator with different filter orders M_sm and M_itp may be implemented. For interpolation relatively few filter coefficients are sufficient, since the performance of an interpolator saturates quickly with increasing filter order. On the other hand, the MSE of a Wiener smoother decreases inversely proportional to the filter dimension.

Appendix

A. Optimality of the separated smoothing and interpolation approach

In order to prove that the SINE approaches the MMSE for unbounded sequence lengths, we transform the Wiener-Hopf equation from (8) into the time domain via an inverse Fourier transform.

Rewriting $\mathbf{R}_{\breve{\mathbf{h}}\breve{\mathbf{h}}}\mathbf{w}_{\mathrm{o}}[n] = \mathbf{r}_{\breve{\mathbf{h}}H}[n]$ in the form of a linear equation system we obtain

$$\sum_{m=0}^{M_{\mathrm{wf}}-1} R_{\breve{H}\breve{H}}[(\tilde{n}-m)D_{\mathrm{f}}] \cdot W_{\mathrm{o}_{m+1}} = R_{HH}[\tilde{n}D_{\mathrm{f}} - \Delta n] \quad (19)$$

with $0 \leq \tilde{n} \leq M_{\mathrm{wf}}-1$, and Δn being defined in (10). Allowing the length of the sequence as well as the filter order to be unbounded, $-\infty < \{m, \tilde{n}\} < \infty$, (19) becomes

$$R_{\breve{H}\breve{H}}[\tilde{n}D_{\mathrm{f}}] * W_{\mathrm{o}_{\tilde{n}}} = R_{HH}[\tilde{n}D_{\mathrm{f}} - \Delta n] , \quad n \in \mathbb{Z} \quad (20)$$

where the $*$ operator represents convolution.

The frequency correlation function at pilot positions, $R_{HH}[nD_{\mathrm{f}}]$ from (3), is obtained by sampling $R_{HH}(f)$ at frequency instants $f = nD_{\mathrm{f}}/T$. Since $R_{HH}(f)$ is a frequency signal, its spectral components are given by an inverse Fourier transform, $r_{hh}(\tau) = \mathcal{F}^{-1}\{R_{HH}(f)\}$, which is the power delay profile (PDP) of the CIR. Transformation of (20) into the time domain yields [14]

$$\left[r_{hh}(\tau) + \tfrac{N_0}{E_s}\right] \cdot w_{\mathrm{o}}(\tau) = r_{hh}(\tau) \cdot \exp(-j2\pi\, \Delta n\, \tau/T) \quad (21)$$
$$\text{with} \quad 0 \leq \tau \leq T/D_{\mathrm{f}}$$

The time domain transfer function $w_{\mathrm{o}}(\tau)$ accounts for the inverse Fourier transform of $\{W_{\mathrm{o}_n}\}$. In the above equation it is assumed that the maximum delay of the channel τ_{max} is smaller than T/D_{f}. The above condition corresponds to the sampling theorem [8]. Then $w_{\mathrm{o}}(\tau)$ can be expressed as

$$w_{\mathrm{o}}(\tau) = \frac{r_{hh}(\tau)}{r_{hh}(\tau) + \frac{N_0}{E_s}} \cdot \exp\left(\tfrac{-j2\pi\, \Delta n\, \tau}{T}\right) \quad (22)$$

Since $r_{hh}(\tau)$ is strictly time limited within the range $\mathcal{T} = [0, T/D_{\mathrm{f}}]$, with $T/D_{\mathrm{f}} \geq \tau_{\mathrm{max}}$, it follows that $w_{\mathrm{o}}(\tau)$ is also zero outside $\mathcal{T}$, for any $N_0 > 0$. Hence, nothing is changed if $w_{\mathrm{o}}(\tau)$ is filtered by an ideal lowpass filter tuned to $\mathcal{T}$:

$$w_{\mathrm{o}}(\tau) = w_{\mathrm{o}}(\tau) \cdot \mathrm{rect}\left(\tfrac{1}{T_{\mathrm{w}}}\left[\tau - \tfrac{1}{2}\right]\right) , \quad \tau_{max} \leq T_{\mathrm{w}} \leq \tfrac{T}{D_{\mathrm{f}}} \quad (23)$$

where the transfer function of an ideal low-pass filter with cut-off $T_{\mathrm{w}}/2$, defined by

$$\mathrm{rect}\left(\tfrac{\tau}{T_{\mathrm{w}}}\right) \triangleq \begin{cases} 1 & |\tau| \leq \frac{T_{\mathrm{w}}}{2} \\ 0 & \text{elsewhere} \end{cases} \quad (24)$$

The next step is to derive the time domain equations for the SINE and show that the result is equivalent to (22). For the smoother the Wiener-Hopf equation is solved without interpolation, so $\Delta n = 0$. According to (21), transforming the Wiener smoother $\mathbf{S}_{\mathrm{o}}$ from (13) into the time domain, the following is obtained

$$s_{\mathrm{o}}(\tau) = \frac{r_{hh}(\tau)}{r_{hh}(\tau) + \frac{N_0}{E_s}} \quad (25)$$

The SINE is a concatenation of two linear filters, which can be expressed by the convolution $S_{\mathrm{o}_n} * V_n = W_n$, which approaches the MMSE if $S_{\mathrm{o}_n} * V_n = W_{\mathrm{o}_n}$. Transformed into the time domain, the convolution is replaced by a multiplication, $w(\tau) = s_{\mathrm{o}}(\tau) \cdot v(\tau)$. Substituting $s_{\mathrm{o}}(\tau) \cdot v(\tau)$ into (22), and also taking (23) into account, the SINE must satisfy

$$s_{\mathrm{o}}(\tau) \cdot v(\tau) = \frac{r_{hh}(\tau)}{r_{hh}(\tau) + \frac{N_0}{E_s}} \exp\left(\tfrac{-j2\pi\, \Delta n\, \tau}{T}\right) \cdot \mathrm{rect}\left(\tfrac{1}{T_{\mathrm{w}}}\left[\tau - \tfrac{1}{2}\right]\right)$$

With the expression for $s_{\mathrm{o}}(\tau)$ from (25), the transfer function of the interpolation filter is in the form

$$v(\tau) = \exp\left(\tfrac{-j2\pi\, \Delta n\, \tau}{T}\right) \cdot \mathrm{rect}\left(\tfrac{1}{T_{\mathrm{w}}}\left[\tau - \tfrac{1}{2}\right]\right) \quad (26)$$

Hence, the SINE approaches the MMSE if $s_{\mathrm{o}}(\tau)$ is a Wiener smoother. Moreover, $v(\tau)$ is an ideal lowpass interpolation filter with one sided passband $[0, T_{\mathrm{w}}]$, $\tau_{max} \leq T_{\mathrm{w}} \leq \frac{T}{D_{\mathrm{f}}}$, and multiplied by a linear increasing phase term.

Finally, if V_n is implemented by a mismatched WIF, V'_n, the remaining problem is to show that $v'(\tau)$ asymptotically approaches (26). The mismatched WIF, V'_n, is obtained by solving (20) for an uniformly distributed PDP, non-zero within $\mathcal{T}$. Hence, in (20) $R_{HH}[\cdot]$ is replaced by $R'_{HH}[\cdot]$ from (11). With the inverse Fourier transform of $R'_{HH}(f)$ given by $r'_{hh}(\tau) = \frac{1}{T_{\mathrm{w}}}\mathrm{rect}\left(\frac{1}{T_{\mathrm{w}}}\left[\tau - \frac{1}{2}\right]\right)$, the mismatched WIF is determined by substituting $r_{hh}(\tau)$ with $r'_{hh}(\tau)$ in (22):

$$v'(\tau) = \begin{cases} \frac{1/T_{\mathrm{w}}}{1/T_{\mathrm{w}} + \varepsilon} \cdot \exp\left(\frac{-j2\pi\, \Delta n\, \tau}{T}\right) & 0 \leq \tau \leq T_{\mathrm{w}} \\ 0 & \text{elsewhere} \end{cases} \quad (27)$$

where ε is a small possible constant which was introduced in section IV-B to maintain numerical stability of the interpolator. Thus, $v'(\tau)$ approaches (26) if $\varepsilon \to 0$.

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