Game Theory

Games in Extensive form & Multistage Games

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Outline

Introduction

- The strategic form of a game is a compact way of describing the mathematical aspects of a game.
- The flavor of many games is lost in such a simple model!
- Three new concepts make their appearance in the extensive form of a game:
  1. The Game Tree
  2. Chance Moves
  3. Information Set

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Definitions

Directed Graph:

A **directed graph** is a pair (T, F) where T is a nonempty set of vertices and F is a function that gives for each $x \in T$ a subset $F(x)$ of T called the followers of x.

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Path:

A **path** from a vertex t_0 to a vertex t_1 is a sequence $x_0, x_1, \cdots, x_n$ of vertices such that $x_0 = t_0, x_n = t_1$ and x_i is a follower of x_{i-1} for $i = 1, \cdots, n$.

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Tree:

A **tree** is a directed graph (T, F) in which there is a special vertex t_0, called the root or the initial vertex, such that for every other vertex $t \in T$there is a unique path beginning at t_0 and ending at t.

The Game Tree

- A play starts at the initial vertex and continues along one of the path eventually ending in one of the terminal vertices.
- At terminal vertices, the rules of the game specify the payoff.
- For $n-$person games, this would an $n-$tuple of payoffs.
- For the non-terminal vertices, there are three possibilities:
  1. The terminal is either assigned to player I who is to choose the move at that position,
  2. Or it is assigned to player II,
  3. Or it may be singled out as a position from which a chance move is made.

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Chance Moves

- Many games involve chance moves.
- Examples: rolling of dice in backgammon or monoplay, dealing cards in bridge or poker
- These chance moves are denoted by N as the moves that chosen by nature.
- It is assumed that the players are aware of the probabilities of the various outcomes resulting from a chance move.

Information Set

- The information set corresponds to the amount of information available to the players about the past moves of the game.
- Not every set of vertices can form an information set.
- In order for a player not to be aware of which vertex of a given information set the game has come to, each vertex in that information set must have the same number of edges leaving it.
- Also the edges from each vertex of an information set, should correspond to the same set of choices.

Kuhn Tree

Definition:

A finite two-person zero-sum game in extensive form is given by:

1. a finite tree with vertices T,
2. a payoff function that assigns a real number to each terminal vertex,
3. a set T_0 of non-empty vertices (representing positions at which chance moves occur) and for each $t \in T_0$, a probability distribution on the edge leading from t,
4. a partition of the rest of the vertices into two groups of information sets $T_{11}, T_{12}, \cdots, T_{1k_1}$ and $T_{21}, T_{22}, \cdots, T_{2k_2}$,
5. for each information set T_{jk} for player j, a set of labels L_{jk}, and for each $t \in T_{jk}$, a one-to-one mapping of L_{jk} onto the set of edges leading from t.

Pure Strategies

- We first should find X and Y, the sets of pure strategies of the players.
- Let $T_{11}, T_{12}, \cdots, T_{1k_1}$ be the information sets for player I and let $L_{11}, L_{12}, \cdots, L_{1k_1}$ be the corresponding set of labels.
- A pure strategy for player I is a rule that tells him exactly what move to make in each of his information sets. Hence,

Pure Strategies

A pure strategy for player I is a k_1 tuple $x = (x_1, \cdots, x_{k_1})$, where for each i, x_i is one of the elements of L_{1i}.

- If there are m_i elements in L_{1i}, the number of pure strategies for player I is the product $m_1 m_2 \cdots m_k$.

Strategic Form

We first define pure strategies X and Y for each player as mentioned before.

Conversion:

If for fixed pure strategies of the players, $x \in X$ and $y \in Y$, the payoff is a random quantity, we replace the payoff by the average value and denote this average value by $A(x, y)$.

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Extensive Form

- The game in strategic form is described by a triplet $< X, Y, A >$ as mentioned before.
- In the strategic form of a game, the players make their choices simultaneously.
- While in the extensive form of a game simultaneous moves are not allowed.
- Simultaneous moves may be made sequentially as follows:
  - We let one of the players, say player I, move first.
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Game of Perfect Information

Definition

A game of perfect information is a game in extensive form in which each information set of every player contains a single vertex.

- ⇒ Each player knows all the past moves of the game including the chance ones!
- ⇒ Every game of perfect information when reduced to strategic form has a saddle point.
- ⇒ Both players have optimal pure strategies.
- ⇒ The saddle point can be found by removing dominated rows and columns.

Chess is a good example!!

Multistage Games

- In these games after a given number of moves (including also chance moves), both players are (simultaneously) given perfect information (which will not be forgotten).
- The general pattern will be:
  - First player I moves;
  - then Player II moves (in ignorance of I's previous move);
  - next a chance move may be made.
  - now both players are given perfect information.
  - These cycles are called a **stage** of the game.
- Depending on the rules of the game, it may be ended or continued to another stage.

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Games of Exhaustion

- Consider the game with matrix Γ of the form:

$$\Gamma = \begin{bmatrix} a_{11} & \Gamma_1 \\ \Gamma_2 & a_{22} \end{bmatrix}$$

where Γ_1 and Γ_2 represent the obligation to play two other games with the respective matrices:

$$\Gamma_1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

- We solve these games by working backward. (treating each stage as a separate game)

$$\Gamma = \begin{bmatrix} a_{11} & v_1 \\ v_2 & a_{22} \end{bmatrix}$$

Example: The Inspection Game

- The matrix for the first stage is:

$$\Gamma_N = \begin{bmatrix} -1 & 1 \\ 1 & \Gamma_{N-1} \end{bmatrix}$$

- Let v_i denotes the value of the game in stage i. Clearly $v_i < 1$ and the game does not have a saddle point. Therefore by applying *equalizing strategy* we get $v_N = \frac{1+v_{N-1}}{3-v_{N-1}}$
- With initial condition $v_1 = 0$, at each stage the value and the optimal strategies for the players are:

$$v_N = \frac{N-1}{N+1} \quad , \quad \left(\frac{1}{N+1}, \frac{N}{N+1}\right)$$

Stochastic Games

- Like games of exhaustion, the stochastic games consist of other games as components.
- Unlike games of exhaustion, there is the possibility for the game to revert to a previous positions.
- This means that the game may consist of original game!
- The game can theoretically continue indefinitely.
- Moreover, there is usually a payoff after each stage of the game, so that, once again an infinite payoff is theoretically possible.
- However, there is a randomization involved which assures the finite play.

Stochastic Games

Definition

A stochastic game is a set of K "game elements" or positions Γ_k. Each game elements is represented by an $m_k \times n_k$ matrix, whose entries are of the form

$$\alpha_{ij}^{k} = a_{ij}^{k} + \sum_{l=1}^{K} \mathbf{P}_{ij}^{kl} \Gamma_l$$

with

$$\mathbf{P}_{ij}^{kl} \geq 0,$$

$$\sum_{l=1}^{K} \mathbf{P}_{ij}^{kl} < 1$$

Example

- As a very simple example, consider the following stochastic game with only one state Γ:

$$\Gamma = \begin{bmatrix} 1 + (3/5)\Gamma & 3 + (1/5)\Gamma \\ 1 + (4/5)\Gamma & 2 + (2/5)\Gamma \end{bmatrix}$$

- Let v be the value of the game,

$$v = \mathrm{Val}\begin{bmatrix} 1 + (3/5)v & 3 + (1/5)v \\ 1 + (4/5)v & 2 + (2/5)v \end{bmatrix}$$

- Clearly v is positive. We can now solve the game:

$$v = \frac{5}{\sqrt{2}}, \qquad p = [0.414, 0.586] \qquad q = [0.293, 0.707]$$

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Recursive Games

- Like games of exhaustion and stochastic games, recursive games consist of other games as components.
- Unlike stochastic games, the termination probabilities of a stage may be zero.
- The game can theoretically continue indefinitely. Also we may have infinite payoff.
- To keep values finite, it is then, generally agreed that there is a payoff only when the game terminates.
- There may also be a payoff a_∞ stipulated in case of unending play.

Recursive Games

Definition

A recursive game consists of a finite set of "game elements" or positions Γ_k represented by matrices A_k whose entries are of the form

$$\alpha_{ij}^k = \mathbf{P}_{ij}^{k0} a_{ij}^k + \sum_{l=1}^{K} \mathbf{P}_{ij}^{kl} \Gamma_l$$

with

$$\mathbf{P}_{ij}^{kl} \geq 0,$$

$$\sum_{l=0}^{K} \mathbf{P}_{ij}^{kl} = 1$$

ϵ-Optimal Strategy

- Consider the recursive game Γ,

$$\Gamma = \begin{bmatrix} \Gamma & 1 \\ 1 & 0 \end{bmatrix}, Q$$

- We want to show that the value of Γ exists and is equal to 1 no matter what the value of the number Q is.
- Player II can restrict her losses to at most 1 by choosing the second column.
- If $Q \geq 1$, player I can guarantee winning at least 1 by playing his first row forever.
- So for $Q \geq 1$, the value of the game is 1 and the optimal strategies for the players are $p = [1, 0]$ and $q = [0, 1]$.
- If $Q < 1$, there is no such an optimal strategy for player I that guarantees him at least 1.

ϵ-Optimal Strategy Cont'

- However for any $\epsilon > 0$ there is a strategy for I that guarantees him an average gain of at least $1 - \epsilon$.

definition

ϵ-optimal strategy is the strategy that guarantees a player an average payoff within ϵ of the value.

- In this case if player I continually uses the mixed strategy $p = [1 - \epsilon, \epsilon]$, he insures that he will eventually choose row 2, so the payoff is either 0 or 1.
- The best thing player II can do then is to choose her second column hoping player I chooses his second row!
- So the expected payoff is $1 \times (1 - \epsilon) + 0 \times \epsilon = 1 - \epsilon$

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THANK YOU