Parameterization of the MISO Interference Channel with Transmit Beamforming and Partial Channel State Information

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***Abstract*—We study the achievable rate region of the multiple-input single-output (MISO) interference channel (IFC), under the assumption that all receivers treat the interference as additive Gaussian noise. The main result is a parameterization of the Pareto boundary for the case where there are two users, the transmitters use beamforming, and the channel state information (CSI) is only partially known at the transmitters. The result is illustrated by two numerical results.**

I. INTRODUCTION

We are concerned with the scenario where we have two independent but mutually interfering wireless systems operating in the same spectral band. System i consists of one base station BS$_i$ that wants to transmit information to a mobile MS$_i$, $i = 1, 2$. The communication in the systems is taking place simultaneously on the same channel. The two mobiles hear a superposition of the signals transmitted from the two base stations. This setup is recognized as an interference channel (IFC) [1]–[3]. We consider the case when BS$_1$ and BS$_2$ have n transmit antennas and MS$_1$ and MS$_2$ have a single receive antenna each. This setup is a multiple-input single-output (MISO) IFC [4]. See Figure 1.

We assume that the transmission consists of scalar coding followed by beamforming and that all propagation channels are frequency-flat. The model for the matched-filtered, symbol-sampled complex baseband data received at MS$_1$ and MS$_2$ will then be

$$\begin{aligned} y_1 &= \boldsymbol{h}_{11}^T \boldsymbol{w}_1 s_1 + \boldsymbol{h}_{21}^T \boldsymbol{w}_2 s_2 + e_1 \\ y_2 &= \boldsymbol{h}_{22}^T \boldsymbol{w}_2 s_2 + \boldsymbol{h}_{12}^T \boldsymbol{w}_1 s_1 + e_2. \end{aligned} \tag{1}$$

In (1) s_1 and s_2 are the transmitted symbols, $\boldsymbol{h}_{ij}$ is the $n \times 1$ channel-vector between BS$_i$ and MS$_j$ and $\boldsymbol{w}_i$ is the beamforming vector used by BS$_i$. Also e_1 and e_2 are noise variables which we model as i.i.d. Gaussian with zero mean and variance σ^2. We model the channel vectors as

$$\boldsymbol{h}_{ij} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{Q}_{ij})$$

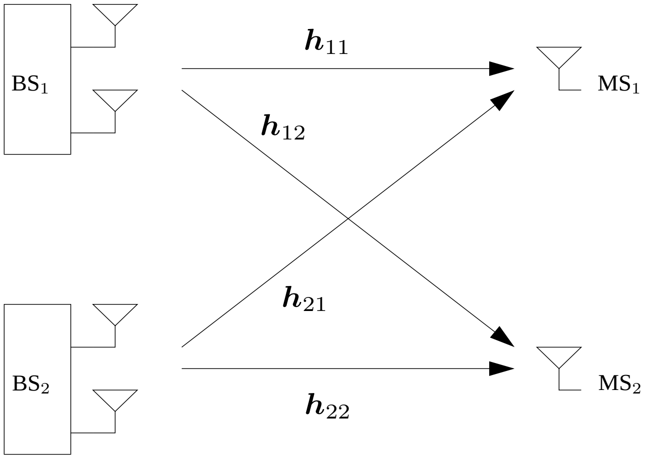

Fig. 1. The two-user MISO interference channel under study (illustrated for $n = 2$ transmit antennas).

that is, $\boldsymbol{h}_{ij}$ is a complex circularly symmetric Gaussian random vector with covariance matrix $\boldsymbol{Q}_{ij}$. Each base station can use transmit power P. Without loss of generality, we shall take $P = 1$. This gives the power constraint

$$\|\boldsymbol{w}_i\|^2 \leq 1, \ \ i = 1, 2.$$

The signal-to-noise ratio (SNR) will be defined as $1/\sigma^2$.

One fundamental difficulty with the MISO IFC is the following [5]. If the two base stations operate in an uncoordinated manner, how should they the choose their beamforming vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$? Obviously there is a conflict situation associated with this choice. A vector $\boldsymbol{w}_1$ which is good for the link BS$_1$ →MS$_1$ may generate substantial interference for MS$_2$ and vice versa.

The IFC is important because it models the situation where a number of unrelated senders (base stations) try to communicate information to different receivers (mobile stations) via a common channel. Recently there is a huge interest in understanding IFCs [6], [7]. Finding the capacity region for general IFCs is still an open problem, but various achievable

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This work was supported in part by the Swedish Research Council (VR) and the Swedish Foundation of Strategic Reasearch (SSF). E. Larsson is a Royal Swedish Academy of Sciences Research (KVA) Fellow supported by a grant from the Knut and Alice Wallenberg Foundation.

rate regions are known. We desire to understand what the achievable rate region looks like in the case that the receiver treats interference as noise. In particular we are interested in the so-called Pareto boundary of the region. This boundary consists of Pareto optimal operating points which are points where it is impossible to improve the rate of one communication link without simultaneously decreasing the rate of the other link. See Definition 1 in Section II below.

Our main result in this paper is a parameterization of the Pareto boundary for the MISO IFC under the assumption that the channel is zero-mean Gaussian with known covariance, that is the transmitter has only statistical channel knowledge. This extends our work in [8] where a corresponding parameterization was presented for the case of complete channel state information (CSI), i.e. $\boldsymbol{h}_{ij}$ deterministic. Here we restrict our treatment to the case that the transmitter performs beamforming (single-stream) transmission, although this is suboptimal for MISO channels with partial channel information [9].

The MISO IFC is important owing to the benefit from multiple antennas compared to the single-input single-output (SISO) scenario. At the same time, the receiver structure is less complex than for multiple-input multiple-output (MIMO) systems, at least when the interference is treated as noise.

Notation: $(\cdot)^*$: complex conjugate. $(\cdot)^T$: transpose. $(\cdot)^H$: Hermitian transpose. $\boldsymbol{I}$: the identity matrix. $\boldsymbol{\Pi}_{\boldsymbol{X}} \triangleq \boldsymbol{X}(\boldsymbol{X}^H\boldsymbol{X})^{-1}\boldsymbol{X}^H$: orthogonal projection onto the column space of $\boldsymbol{X}$. $\boldsymbol{\Pi}_{\boldsymbol{X}}^{\perp} \triangleq \boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{X}}$: orthogonal projection onto the orthogonal complement of the column space of $\boldsymbol{X}$.

II. THE PARETO BOUNDARY

For fixed channel-vectors $\{\boldsymbol{h}_{ij}\}$ and a given pair of beamforming vectors $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$, the following rates are achievable:

$$R_1 = \log_2\left(1 + \frac{|\boldsymbol{w}_1^T\boldsymbol{h}_{11}|^2}{|\boldsymbol{w}_2^T\boldsymbol{h}_{21}|^2 + \sigma^2}\right) \tag{2}$$

for the link BS$_1$ →MS$_1$, and

$$R_2 = \log_2\left(1 + \frac{|\boldsymbol{w}_2^T\boldsymbol{h}_{22}|^2}{|\boldsymbol{w}_1^T\boldsymbol{h}_{12}|^2 + \sigma^2}\right) \tag{3}$$

for BS$_2$ →MS$_2$. We define the achievable rate region as

$$\mathcal{R} = \bigcup_{\boldsymbol{w}_1,\boldsymbol{w}_2,\|\boldsymbol{w}_i\|^2\le 1} (R_1, R_2).$$

Since we only know the covariance matrices of the channel-vectors we will study the expected value of R_i. Define $\bar{R}_i$ which is the expected value of R_i:

$$\bar{R}_i = E_{\boldsymbol{h}_{11},\boldsymbol{h}_{21}}[R_1|\boldsymbol{h}_{11},\boldsymbol{h}_{21}], \tag{4}$$

where the expectation is over the distribution of $\boldsymbol{h}_{ij}$. Now we will consider the region

$$\bar{\mathcal{R}} = \bigcup_{\boldsymbol{w}_1,\boldsymbol{w}_2,\|\boldsymbol{w}_i\|^2\le 1} (\bar{R}_1, \bar{R}_2).$$

We will need the following definition of Pareto optimality.

Definition 1 *A rate tuple ($\bar{R}_1$,$\bar{R}_2$) is Pareto optimal if there is no other tuple ($\bar{Q}_1$,$\bar{Q}_2$) with ($\bar{Q}_1$,$\bar{Q}_2$)> ($\bar{R}_1$,$\bar{R}_2$) and ($\bar{Q}_1$,$\bar{Q}_2$) $\neq$ ($\bar{R}_1$,$\bar{R}_2$). (The inequality is component-wise.)*

In order to calculate $\bar{R}_i$ note that

$$\boldsymbol{w}_i^T\boldsymbol{h}_{ij} \sim \mathcal{CN}(0, 1/\lambda_{ij}),$$

where

$$\lambda_{ij} \triangleq \frac{1}{\boldsymbol{w}_i^T\boldsymbol{Q}_{ij}\boldsymbol{w}_i^*} = \frac{1}{\left\|\boldsymbol{Q}_{ij}^{1/2}\boldsymbol{w}_i^*\right\|^2} \tag{5}$$

Therefore

$$\alpha_{ij} \triangleq |\boldsymbol{w}_i^T\boldsymbol{h}_{ij}|^2 \sim \exp(\lambda_{ij}),$$

that is, $p_{\alpha_{ij}}(x) = \lambda_{ij}e^{-\lambda_{ij}x},\ x \ge 0$. By evaluating equation (4) for $i = 1$ the expected value of R_1 can be expressed as

$$\begin{aligned}
\bar{R}_1 &= E_{\boldsymbol{h}_{11},\boldsymbol{h}_{21}}[R_1|\boldsymbol{h}_{11},\boldsymbol{h}_{21}] \\
&= E_{\boldsymbol{h}_{11},\boldsymbol{h}_{21}}\left[\log_2\left(1 + \frac{|\boldsymbol{w}_1^T\boldsymbol{h}_{11}|^2}{|\boldsymbol{w}_2^T\boldsymbol{h}_{21}|^2 + \sigma^2}\right)\right] \\
&= E_{\alpha_{11},\alpha_{21}}[R_1|\alpha_{11},\alpha_{21}] \\
&= E_{\alpha_{11},\alpha_2}\left[\log_2\left(1 + \frac{\alpha_{11}}{\alpha_{21} + \sigma^2}\right)\right] \\
&= \int_0^\infty\int_0^\infty p(\alpha_{11})p(\alpha_{21}) \cdot \\
&\quad \cdot \log_2\left(1 + \frac{\alpha_{11}}{\alpha_{21} + \sigma^2}\right) d\alpha_{11}d\alpha_{21} \\
&= \int_0^\infty\int_0^\infty \lambda_{11}e^{-\lambda_{11}\alpha_{11}}\lambda_{21}e^{-\lambda_{21}\alpha_{21}} \cdot \\
&\quad \cdot \log_2\left(1 + \frac{\alpha_{11}}{\alpha_{21} + \sigma^2}\right) d\alpha_{11}d\alpha_{21}
\end{aligned} \tag{6}$$

where we used that α_{11} and α_{21} are independent since $\boldsymbol{h}_{11}$ and $\boldsymbol{h}_{21}$ are independent. The expression for $\bar{R}_2$ is similar as for $\bar{R}_1$.

Proposition 1: *Suppose the vectors $\boldsymbol{w}_i,\ i = 1, 2$ correspond to a rate point ($\bar{R}_1, \bar{R}_2$) on the Pareto boundary. Then*

a) $\boldsymbol{w}_i \in$ span$\{\boldsymbol{Q}_{i1}^*, \boldsymbol{Q}_{i2}^*\}$[1] and

b) $\|\boldsymbol{w}_i\| = 1$.

Item b) in Proposition 1 means that both base stations use full power. In order to prove Proposition 1 we first state and prove the following Lemma.

Lemma 1: *Let λ_{ij} be defined according (5). Then $\bar{R}_1$ is monotonously decreasing with λ_{11} for fixed λ_{21} and monotonously increasing with λ_{21} for fixed λ_{11}.*

Lemma 1 is proved in the appendix.

Proof (Proposition 1): We give the proof for $\boldsymbol{w}_1$; the proof for $\boldsymbol{w}_2$ goes in a similar manner. The proof is by contradiction.

a) In order to arrive at a contradiction, suppose the statement in the proposition is false. Then there exists a $\boldsymbol{w}_1$, $\|\boldsymbol{w}_1\| \le 1$, that corresponds to a rate point on the boundary but for which $\boldsymbol{w}_1 \notin$ span$\{\boldsymbol{Q}_{i1}^*, \boldsymbol{Q}_{i2}^*\}$. Then we can write:

$$\boldsymbol{w}_1 = \sum_{i=1}^{K}\alpha_{1i}\boldsymbol{y}_{1i} + \sum_{j=1}^{K'}\beta_{1j}\boldsymbol{z}_{1j} + \sum_{m=1}^{n-K-K'}\gamma_{1m}\boldsymbol{u}_{1m}$$

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[1] span$\{A, B\}$ is shorthand for span$\{A\} \cup$ span$\{B\}$.

for some $\{\alpha_{1i}\}$, $\{\beta_{1i}\}$ and $\{\gamma_{1i}\}$, where

$$\begin{aligned} K \, &\triangleq \text{rank}\{\boldsymbol{Q}_{11}^*\}, \\ K' &\triangleq \text{rank}\{\boldsymbol{Q}_{12}^*\} \\ K'' &\triangleq \text{rank}\{\boldsymbol{Q}_{11}^*\boldsymbol{Q}_{12}^*\}^{\perp} \end{aligned}$$

and

$$\begin{array}{ll} \{\boldsymbol{y}_{1i}\}_{i=1}^{K} & \text{is an ON-basis for span}\{\boldsymbol{Q}_{11}^*\} \\ \{\boldsymbol{z}_{1j}\}_{j=1}^{K'} & \text{is an ON-basis for span}\{\boldsymbol{Q}_{12}^*\} \\ \{\boldsymbol{u}_{1m}\}_{m=1}^{K''} & \text{is an ON-basis for span}\{\boldsymbol{Q}_{11}^*, \boldsymbol{Q}_{12}^*\}^{\perp}. \end{array}$$

Notice that not all vectors $\{\boldsymbol{y}_{1i}, \boldsymbol{z}_{1j}\}$ necessarily need to be orthogonal. If the statement in a) is false, then $\gamma_{1m} \neq 0$ for some m, say $m = m'$.

Let

$$\boldsymbol{w}_1' \triangleq \boldsymbol{w}_1 - \gamma_{1m'}\boldsymbol{u}_{1m'}.$$

We shall show that for fixed $\boldsymbol{w}_2$ we have

i) $\bar{R}_1' = \bar{R}_1$

ii) $\bar{R}_2' = \bar{R}_2$

iii) $\|\boldsymbol{w}_1'\| < \|\boldsymbol{w}_1\| \leq 1$

where $(\bar{R}_1', \bar{R}_2')$ is the rate point associated with $(\boldsymbol{w}_1', \boldsymbol{w}_2)$, that is

$$\bar{R}_1' \triangleq E_{\boldsymbol{h}_{11},\boldsymbol{h}_{21}}\left[\log_2\left(1 + \frac{|\boldsymbol{w}_1'^T\boldsymbol{h}_{11}|^2}{|\boldsymbol{w}_2^T\boldsymbol{h}_{21}|^2 + \sigma^2}\right)\right]$$

and

$$\bar{R}_2' \triangleq E_{\boldsymbol{h}_{11},\boldsymbol{h}_{21}}\left[\log_2\left(1 + \frac{|\boldsymbol{w}_2^T\boldsymbol{h}_{22}|^2}{|\boldsymbol{w}_1'^T\boldsymbol{h}_{12}|^2 + \sigma^2}\right)\right].$$

Item *i)* follows because $\boldsymbol{u}_{1m'} \perp \boldsymbol{h}_{11}^*$:

$$|\boldsymbol{w}_1'^T\boldsymbol{h}_{11}|^2 = |\boldsymbol{w}_1^T\boldsymbol{h}_{11} - \gamma_{1m'}\boldsymbol{u}_{1m'}^T\boldsymbol{h}_{11}|^2 = |\boldsymbol{w}_1^T\boldsymbol{h}_{11}|^2$$

Item *ii)* follows because $\boldsymbol{u}_{1m'} \perp \boldsymbol{h}_{12}^*$:

$$|\boldsymbol{w}_1'^T\boldsymbol{h}_{12}|^2 = |\boldsymbol{w}_1^T\boldsymbol{h}_{12} - \gamma_{1m'}\boldsymbol{u}_{1m'}^T\boldsymbol{h}_{12}|^2 = |\boldsymbol{w}_1^T\boldsymbol{h}_{12}|^2$$

To see the third item, consider

$$\begin{aligned} \|\boldsymbol{w}_1\|^2 &= \left\|\boldsymbol{\Pi}_{\text{span}\{\boldsymbol{Q}_{11}^*,\boldsymbol{Q}_{12}^*\}}\boldsymbol{w}_1\right\|^2 + \\ &\quad + \left\|\boldsymbol{\Pi}_{\text{span}\{\boldsymbol{Q}_{11}^*,\boldsymbol{Q}_{12}^*\}}^{\perp}\boldsymbol{w}_1\right\|^2 \\ &= \left\|\boldsymbol{\Pi}_{\text{span}\{\boldsymbol{Q}_{11}^*,\boldsymbol{Q}_{12}^*\}}\boldsymbol{w}_1\right\|^2 + \sum_{m=1}^{K''}|\gamma_{1m}|^2. \end{aligned}$$

This implies that

$$\|\boldsymbol{w}_1'\|^2 = \|\boldsymbol{w}_1\|^2 - |\gamma_{1m'}|^2 < \|\boldsymbol{w}_1\|^2,$$

which shows *iii)*.

Next define, for given $\boldsymbol{\delta}$

$$\boldsymbol{w}_1'' \triangleq \boldsymbol{w}_1' + \boldsymbol{\delta}.$$

Also let

$$\bar{R}_1'' \triangleq E\left[\log_2\left(1 + \frac{|\boldsymbol{w}_1''^T\boldsymbol{h}_{11}|^2}{|\boldsymbol{w}_2^T\boldsymbol{h}_{21}|^2 + \sigma^2}\right)\right]$$

and

$$\bar{R}_2'' \triangleq E\left[\log_2\left(1 + \frac{|\boldsymbol{w}_2^T\boldsymbol{h}_{22}|^2}{|\boldsymbol{w}_1''^T\boldsymbol{h}_{12}|^2 + \sigma^2}\right)\right].$$

We will now show that there exists a $\boldsymbol{\delta}$ such that $\bar{R}_1'' > \bar{R}_1' = \bar{R}_1$, $\bar{R}_2'' = \bar{R}_2$ and $\|\boldsymbol{w}_1''\| \leq 1$ that is,

iv) $\boldsymbol{w}_1''$ will cause an increase in $\bar{R}_1$ compared to $\boldsymbol{w}_1$.

v) $\bar{R}_2$ will be unchanged when $\boldsymbol{w}_1''$ is used instead of $\boldsymbol{w}_1$.

vi) $\boldsymbol{w}_1''$ satisfies the power constraint.

We will, according to Lemma 1, decrease λ_{11} in (5), which will cause an increase of $\bar{R}_1$. Item *iv)* is satisfied if

$$\begin{aligned} \boldsymbol{w''}_1^T\boldsymbol{Q}_{11}\boldsymbol{w''}_1^* &= (\boldsymbol{w}_1' + \boldsymbol{\delta})^T\boldsymbol{Q}_{11}(\boldsymbol{w}_1' + \boldsymbol{\delta})^* \\ &\overset{(*)}{=} (\boldsymbol{w}_1 + \boldsymbol{\delta})^T\boldsymbol{Q}_{11}(\boldsymbol{w}_1 + \boldsymbol{\delta})^* \\ &> \boldsymbol{w}_1^T\boldsymbol{Q}_{11}\boldsymbol{w}_1^* \end{aligned} \tag{7}$$

where the equality $(*)$ is due to the fact that $\boldsymbol{u}_{1m'} \perp \boldsymbol{h}_{11}^*$ with probability 1 (cf. *i)* above). The inequality in (7) is satisfied if $\boldsymbol{\delta}$ is chosen such that

$$2\text{Re}\{\boldsymbol{\delta}^T\boldsymbol{Q}_{11}\boldsymbol{w}_1^*\} > -\boldsymbol{\delta}^T\boldsymbol{Q}_{11}\boldsymbol{\delta}^*. \tag{8}$$

Next, note that item *v)* is satisfied if

$$\boldsymbol{Q}_{12}^*\boldsymbol{\delta} = \boldsymbol{0} \Leftrightarrow \boldsymbol{\delta} \in \text{span}\{\boldsymbol{Q}_{12}^*\}^{\perp}. \tag{9}$$

To construct $\boldsymbol{\delta}$ we proceed as follows. First choose a $\bar{\boldsymbol{\delta}}$ so that equation (9) is satisfied and such that $\left\|\bar{\boldsymbol{\delta}}\right\| = 1$. One way to do this is to solve

$$\begin{cases} \boldsymbol{Q}_{12}^*\tilde{\boldsymbol{\delta}} = \boldsymbol{0} & (\text{rank}\{\boldsymbol{Q}_{12}^*\} \text{ equations}) \\ \boldsymbol{\Psi}\tilde{\boldsymbol{\delta}} = \boldsymbol{0} & (\text{rank}\{\boldsymbol{Q}_{11}^*, \boldsymbol{Q}_{12}^*\}^{\perp} \text{ equations}) \end{cases} \tag{10}$$

where the columns in the matrix $\boldsymbol{\Psi}$ is an ON-basis for span$\{\boldsymbol{Q}_{11}^*, \boldsymbol{Q}_{12}^*\}^{\perp}$. The problem has n dimensions. If span$\{\boldsymbol{Q}_{11}^*\}$ = span$\{\boldsymbol{Q}_{12}^*\}$, then we will have n equations in the equation system (10) and we cannot find any $\tilde{\boldsymbol{\delta}} \neq \boldsymbol{0}$ that satisfies (10). But this case is unlikely since this implies that the receivers are so close to each other that the corresponding channel vectors belong to the same subspace. If span$\{\boldsymbol{Q}_{11}^*\} \neq$ span$\{\boldsymbol{Q}_{12}^*\}$, then there need to be at least $n \geq 3$ transmitter antennas to find any $\tilde{\boldsymbol{\delta}} \neq \boldsymbol{0}$.

When a $\tilde{\boldsymbol{\delta}}$ is found, normalize it by setting $\bar{\boldsymbol{\delta}} = \tilde{\boldsymbol{\delta}} / \left\|\tilde{\boldsymbol{\delta}}\right\|$. Then choose $\boldsymbol{\delta} = \epsilon e^{i\phi}\bar{\boldsymbol{\delta}}$ where $\epsilon > 0$ (to be chosen later) and $\phi = -\arg\bar{\boldsymbol{\delta}}^T\boldsymbol{Q}_{11}\boldsymbol{w}_1^*$. This choice will make $2\text{Re}\{\boldsymbol{\delta}^T\boldsymbol{Q}_{11}\boldsymbol{w}_1^*\} > 0$.

It remains to choose $\epsilon > 0$ such that $\|\boldsymbol{w}_1''\|^2 \leq 1$. But

$$\|\boldsymbol{w}_1''\| = \|\boldsymbol{w}_1' + \boldsymbol{\delta}\| \leq \|\boldsymbol{w}_1'\| + \|\boldsymbol{\delta}\| = \|\boldsymbol{w}_1'\| + \epsilon \leq 1.$$

This requires that $\epsilon \leq 1 - \|\boldsymbol{w}_1'\|^2$. For example, take $\epsilon = 1 - \|\boldsymbol{w}_1'\|^2$. Hence we have shown: $(\boldsymbol{w}_1'', \boldsymbol{w}_2)$ achieves $(\bar{R}_1'', \bar{R}_2)$ where $\bar{R}_1'' > \bar{R}_1$, and $\bar{R}_2'' = \bar{R}_2$. Hence, $(\bar{R}_1, \bar{R}_2)$ cannot be on the Pareto boundary so we have a contradiction.

b) To show that we must have $\|\boldsymbol{w}_1\| = 1$ at the boundary assume that $\|\boldsymbol{w}_1\| < 1$. Let $\boldsymbol{w}_1' = \boldsymbol{w}_1 + \boldsymbol{\delta}$ where $\boldsymbol{\delta}$ is chosen according to the recipe above. This shows that if $\|\boldsymbol{w}_1\|^2 < 1$ then it is possible to choose a new $\boldsymbol{w}_1'$ such that $\|\boldsymbol{w}_1'\| = 1$, $\bar{R}_1$ is increased and $\bar{R}_2$ is unchanged. □

III. THE SPECIAL CASE OF RANK$\{\boldsymbol{Q}_{ij}\} = 1$

In this section we consider the special case of rank$\{\boldsymbol{Q}_{ij}\} = 1$. This is the case with no angular spread. When this is the case $\boldsymbol{Q}_{ij}$ can be written as

$$\boldsymbol{Q}_{ij} = \boldsymbol{q}_{ij}\boldsymbol{q}_{ij}^H.$$

where $\boldsymbol{q}_{ij}$ is an n-vector. Define

$$\bar{h}_{ij} \sim \mathcal{CN}(0, 1).$$

Then we can write

$$\boldsymbol{h}_{ij} = \bar{h}_{ij}\boldsymbol{q}_{ij} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{q}_{ij}\boldsymbol{q}_{ij}^H) = \mathcal{CN}(\boldsymbol{0}, \boldsymbol{Q}_{ij}).$$

This means that the channel state information is known up to an unknown scalar constant.

In this special case, the parameterization in Proposition 1 essentially reduces to Proposition 1 in [8].

IV. AN EXPLICIT FORMULA FOR THE AVERAGE RATE

To be able to make simulations we need an explicit formula for the average rates $\bar{R}_i,\ \ i = 1, 2$. First we rewrite (6) as

$$\begin{aligned}
\bar{R}_1 &= E_{\alpha_{11},\alpha_{21}}\left[\log_2\left(1 + \frac{\alpha_{11}}{\alpha_{21} + \sigma^2}\right)\right] \\
&= E_{\alpha_{11},\alpha_{21}}\left[\log_2(\sigma^2 + \alpha_{11} + \alpha_{21}) - \log_2(\sigma^2 + \alpha_{21})\right] \\
&= E_{\alpha_{11},\alpha_{21}}\left[\log_2(\sigma^2 + \alpha_{11} + \alpha_{21})\right] \\
&\quad - E_{\alpha_{21}}\left[\log_2(\sigma^2 + \alpha_{21})\right] \\
&= E_{\bar{\alpha}_{11}\bar{\alpha}_{21}}\left[\log_2\left(\sigma^2 + \frac{\bar{\alpha}_{11}}{\lambda_{11}} + \frac{\bar{\alpha}_{21}}{\lambda_{21}}\right)\right] \\
&\quad - E_{\bar{\alpha}_{21}}\left[\log_2\left(\sigma^2 + \frac{\bar{\alpha}_{21}}{\lambda_{21}}\right)\right]
\end{aligned}$$

where $\bar{\alpha}_{11}$ and $\bar{\alpha}_{21}$ are independent standard exponentially distributed random variables. We then define

$$\begin{aligned}
p_1 &\triangleq 1/\lambda_{11} = \left\|\boldsymbol{Q}_{11}^{1/2}\boldsymbol{w}_1^*\right\|^2, \quad p_2 \triangleq 1/\lambda_{21} = \left\|\boldsymbol{Q}_{21}^{1/2}\boldsymbol{w}_2^*\right\|^2 \\
q_1 &\triangleq 1/\lambda_{22} = \left\|\boldsymbol{Q}_{22}^{1/2}\boldsymbol{w}_2^*\right\|^2, \quad q_2 \triangleq 1/\lambda_{12} = \left\|\boldsymbol{Q}_{12}^{1/2}\boldsymbol{w}_1^*\right\|^2
\end{aligned}$$

and

$$f(x) \triangleq e^{\sigma^2/x}E_1(\sigma^2/x),\ \ x > 0.$$

where $E_1(\cdot)$ is the exponential integral [11]:

$$E_1(y) \triangleq \int_y^\infty \frac{e^{-t}}{t}dt.$$

From (37) in [10] we then have

$$\begin{aligned}
\bar{R}_1 &= \lambda_{11}\lambda_{21}\left(\frac{\exp(\sigma^2\lambda_{11})E_1(\sigma^2\lambda_{11})/\lambda_{11}}{\lambda_{21} - \lambda_{11}}\right. \\
&\quad \left. + \frac{\exp(\sigma^2\lambda_{21})E_1(\sigma^2\lambda_{21})/\lambda_{21}}{\lambda_{11} - \lambda_{21}}\right) - \\
&\quad - \exp(\sigma^2\lambda_{21})E_1(\sigma^2\lambda_{21}) \\
&= \frac{p_1}{p_2 - p_1}(f(p_2) - f(p_1)) \qquad (11)
\end{aligned}$$

We also have

$$\bar{R}_2 = \frac{q_1}{q_2 - q_1}(f(q_2) - f(q_1)). \qquad (12)$$

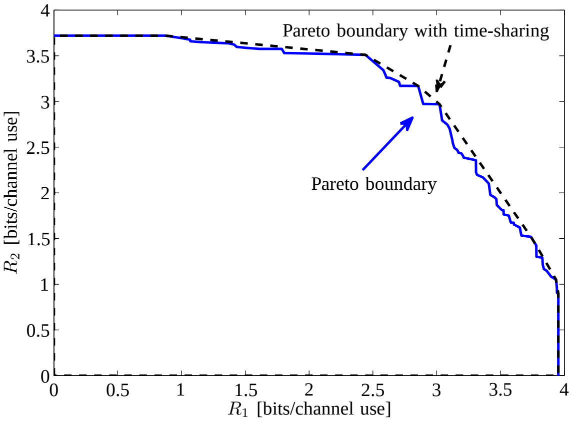

Fig. 2. Rate region for a system with low interference, SNR=10 dB. (The roughness of the curve is an artifact of the numerical simulation.)

V. SIMULATIONS

The purpose of this section is to illustrate Proposition 1 with two numerical examples. We used two different sets of channel covariance matrices: one set with low interference between the systems (see Fig. 2), and one set with high interference (see Fig. 3). For the case with high interference we chose the covariance matrices such that span$\{\boldsymbol{Q}_{12}\}$ was close to span$\{\boldsymbol{Q}_{11}\}$ and such that span$\{\boldsymbol{Q}_{21}\}$ was close to span$\{\boldsymbol{Q}_{22}\}$. In both simulations we used $n = 5$ transmit antennas and rank$\{\boldsymbol{Q}_{ij}\} = 2$.

To plot the Pareto boundary we generated a large number of random beamforming vectors using the parameterization of Proposition 1. More precisely, we constructed the beamforming vectors according to

$$\boldsymbol{w}_i = \sum_{j=1}^{N_i} a_{i,j}\boldsymbol{\psi}_{i,j} \qquad (13)$$

where $N_i = $ rank$\{\boldsymbol{Q}_{i1}^*, \boldsymbol{Q}_{i2}^*\}$ and $\{\boldsymbol{\psi}_{i,j}\}_{j=1}^{N_i}$ is an ON-basis for span$\{\boldsymbol{Q}_{i1}^*, \boldsymbol{Q}_{i2}^*\}$. Also, $a_{i,j}$ are complex-valued numbers, chosen randomly but such that $\sum_{j=1}^{N_i}|a_{i,j}|^2 = 1$ (i.e., such that the power constraint is satisfied). For each simulation we used $5 \cdot 10^{10}$ pairs of beamforming vectors, generated according to (13).

Figs. 2 and 3 show the results for the cases of low interference and high interference, respectively. One observation is that the rate region is not necessarily convex.

VI. TIME-SHARING

When the rate region is non-convex, it is possible to use time-sharing to achieve points outside the Pareto boundary. The idea with time-sharing is that the systems operate at one given rate point a fraction of the time, say τ, and at another rate point the rest of the time (the remaining fraction is $1-\tau$). This means that any point which lies on a straight line between two points in the rate region is achievable by time-sharing. As a consequence, the rate region achievable by time-sharing will

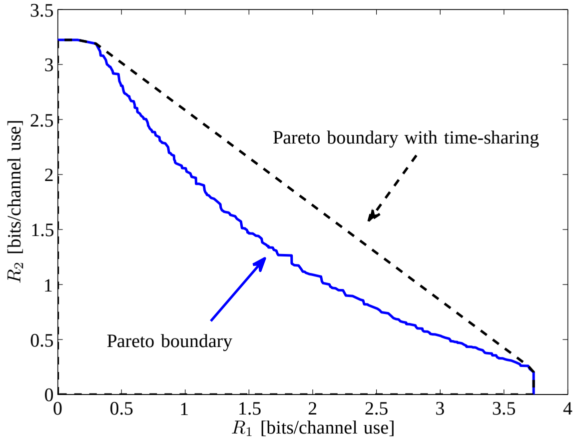

Fig. 3. Rate region for a system with high interference, which leads to a non-convex region. The dashed line is the convex hull. SNR=10 dB.

be the convex hull of the original region. Let $(\bar{R}_1, \bar{R}_2)$ and $(\bar{R}_1', \bar{R}_2')$ be two rate points within the Pareto boundary. The convex hull is then defined as

$$\bar{\mathcal{R}} = \bigcup_{\substack{0 \le \tau \le 1 \\ (\bar{R}_1, \bar{R}_2) \in \mathcal{R} \\ (\bar{R}_1', \bar{R}_2') \in \mathcal{R}}} \left(\tau \bar{R}_1 + (1-\tau)\bar{R}_1', \tau \bar{R}_2 + (1-\tau)\bar{R}_2'\right).$$

This is the dashed line in Fig. 2 and Fig. 3. Proposition 1 can be useful also to parameterize the region with time-sharing.

VII. CONCLUSION

The motivation for this paper has been the huge interest in IFCs as a model for spectrum resource conflicts. We have studied the MISO IFC, and especially the case when the CSI is not perfectly known at the transmitter. Our main contribution is a parameterization of the Pareto boundary of the achievable rate region of the MISO IFC. The results should be useful for future research on resource allocation and spectrum sharing for situations that are well modeled via the MISO IFC.

APPENDIX

Proof of Lemma 1: From (6) we have

$$\begin{aligned} \bar{R}_1 &= \int_0^\infty \int_0^\infty \lambda_1 e^{-\lambda_1 \alpha_1} \lambda_2 e^{-\lambda_2 \alpha_2} \cdot \\ &\quad \cdot \log_2 \left(1 + \frac{\alpha_1}{\alpha_2 + \sigma^2}\right) d\alpha_1 d\alpha_2. \end{aligned}$$

Set

$$\begin{array}{ll} \alpha_1' = \lambda_1 \alpha_1 & d\alpha_1' = \lambda_1 d\alpha_1 \\ \alpha_2' = \lambda_2 \alpha_2 & d\alpha_2' = \lambda_2 d\alpha_2 \end{array}.$$

This implies that

$$\begin{aligned} \bar{R}_1 &= \int_0^\infty \int_0^\infty e^{-\alpha_1'} e^{-\alpha_2'} \cdot \\ &\quad \cdot \log_2 \left(1 + \frac{\alpha_1'/\lambda_1}{\alpha_2'/\lambda_2 + \sigma^2}\right) d\alpha_1' d\alpha_2' \\ &< \int_0^\infty \int_0^\infty e^{-\alpha_1'} e^{-\alpha_2'} \cdot \\ &\quad \cdot \log_2 \left(1 + \frac{\alpha_1'/(\lambda_1 + \Delta_1)}{\alpha_2'/\lambda_2 + \sigma^2}\right) d\alpha_1' d\alpha_2' \end{aligned}$$

for any $\Delta_1 > 0$. This shows "decreasing with λ_1 for fixed λ_2". Similarly to show "increasing with λ_2 for fixed λ_1":

$$\begin{aligned} \bar{R}_1 &= \int_0^\infty \int_0^\infty e^{-\alpha_1'} e^{-\alpha_2'} \cdot \\ &\quad \cdot \log_2 \left(1 + \frac{\alpha_1'/\lambda_1}{\alpha_2'/\lambda_2 + \sigma^2}\right) d\alpha_1' d\alpha_2' \\ &> \int_0^\infty \int_0^\infty e^{-\alpha_1'} e^{-\alpha_2'} \cdot \\ &\quad \cdot \log_2 \left(1 + \frac{\alpha_1'/\lambda_1}{\alpha_2'/(\lambda_2 + \Delta_2) + \sigma^2}\right) d\alpha_1' d\alpha_2' \end{aligned}$$

for any $\Delta_2 > 0$. ☐

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