Digital Filters, 160601 – Solutions

1 a) $H(e^{j\omega T}) = H_R(\omega T)\, e^{j\Theta(\omega T)}$, $|H| = |H_R|$

real-valued zero-phase freq. resp.

b) Ladder WDF – low sensitivity in passb. & stopb.

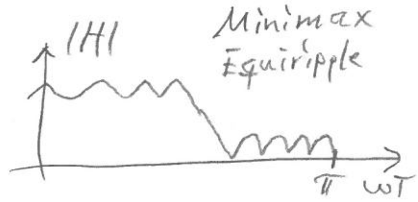
Lattice WDF – low sensit. in passb.

high – " – stopb.

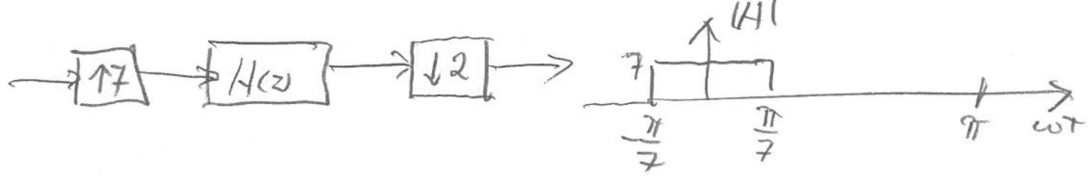
c) recursive – "feedback", nonrecursive – "feed forward"

IIR always recursive, FIR normally nonrecursive
can be realized recursively

d)



e)



2) Spec. of analog HP: $w_{ac} = \frac{2}{T} \tan\left(\frac{\omega_c T}{2}\right) = \frac{2}{T} \cdot 3{,}078$

$w_{as} = \frac{2}{T} \tan\left(\frac{\omega_s T}{2}\right) = \frac{2}{T} \cdot 0{,}680$

Spec of analog LP $\Omega_{ac} = \frac{\omega_T^2}{w_{ac}} = \left| \text{set } \omega_T^2 = w_{ac} \right| = 1$

$\Omega_{as} = \frac{w_{ac}}{w_{as}} = 4{,}5287$

Normalized poles and zeros (Table, $N=3$, $\rho=15\%$, $\theta=13$)

Poles: $-0{,}997065$

$-0{,}466446 \pm j\,1{,}211374$

zeros: $\pm j\,3{,}116621$

∞

Denormalized = Normalized, $\Omega_{ac} = 1$

LP → HP, $s_{HP} = \frac{\omega_T^2}{s_{LP}} \Rightarrow$

poles: $(-3{,}0867) \cdot \frac{T}{2}$

$(-0{,}8520 \pm j\,2{,}2126) \cdot \frac{T}{2}$

zeros $\pm j\,0{,}6015$

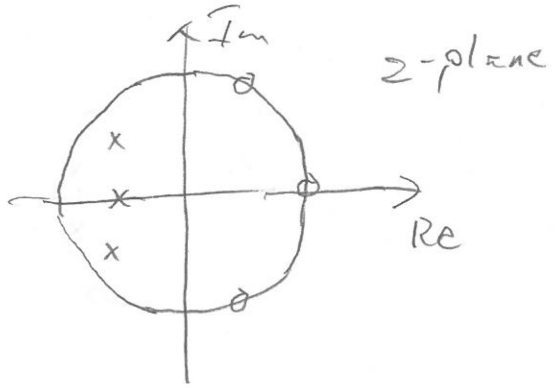
0

Analog to digital, bilinear transf. $z = \frac{1 + s \cdot \frac{T}{2}}{1 - s \cdot \frac{T}{2}} \Rightarrow$

poles: $-0{,}510610$

$-0{,}555101 \pm j\,0{,}531530$

zeros: $0{,}468629 \pm j\,0{,}883395$



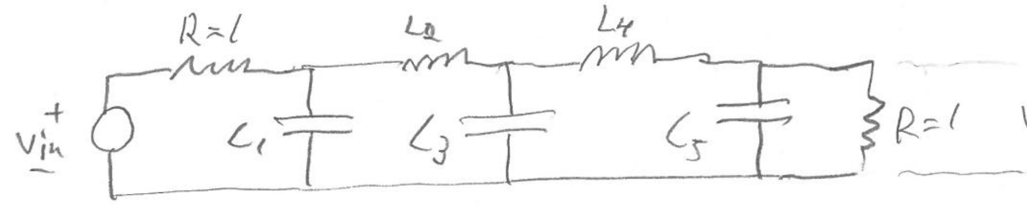
3) Spec of analog LP $\quad \omega_{ac} = \tan\left(\frac{\omega_c T}{2}\right) = 0{,}3249$

$\omega_{as} = \tan\left(\frac{\omega_s T}{2}\right) = 0{,}8541$

Filter order $N = 5$ $\left(Ch.I, \ \frac{\omega_{as}}{\omega_{ac}} = 2{,}63\right)$

π-net

For ex, to keep it as simple as possible (R arbitrary)



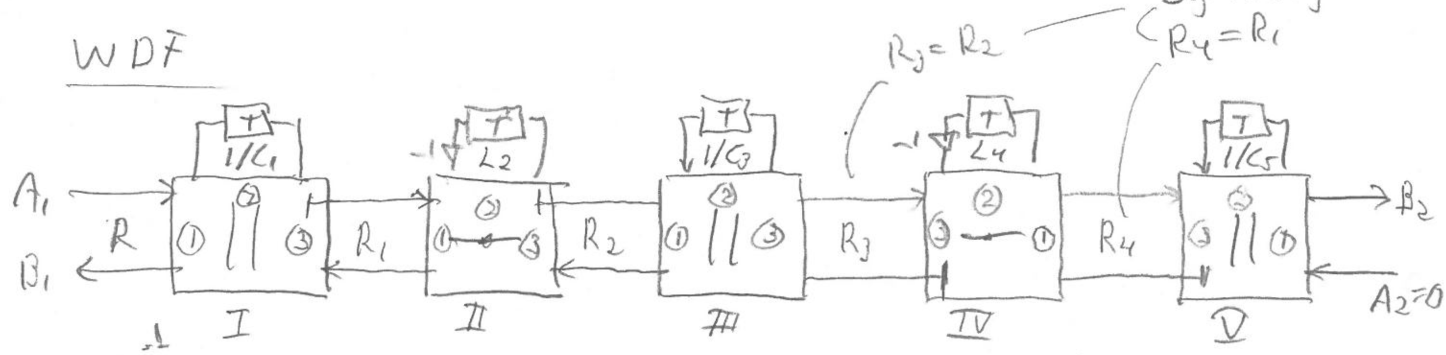
From Table: $\quad C_1' = C_5' = 1{,}1468$, $\quad C_3' = 1{,}9750$

$L_2' = L_4' = 1{,}3712$

Denormalize: mult by $\frac{1}{\omega_{ac}}$ $\quad C_1 = C_5 = 3{,}5294$, $\quad C_3 = 6{,}0788$

$L_2 = L_4 = 4{,}220$

WDF



I: $G_1 = \frac{1}{R_1} = \frac{1}{R} + C_1 = 4{,}5294$

$\alpha_1 = \frac{2 \cdot \frac{1}{R}}{\frac{1}{R} + C_1 + G_1} = 0{,}2208$

$\alpha_2 = 1 - \alpha_1 = 0{,}7792$

$\alpha_3 = 1$

II: $R_2 = R_1 + L_2 = 4{,}4410 = \frac{1}{G_2}$

$\alpha_1 = \frac{2R_1}{R_1 + L_2 + R_2} = 0{,}04971$

$\alpha_2 = 1 - \alpha_1 = 0{,}9503$

$\alpha_3 = 1$

III: $\alpha_1 = \frac{2G_2}{G_2 + C_3 + G_3} = 0{,}06898$, $\quad G_3 = G_2$

$\alpha_2 = 2 - \alpha_1 - \alpha_3 = 2 - 2\alpha_1 = 1{,}8620$

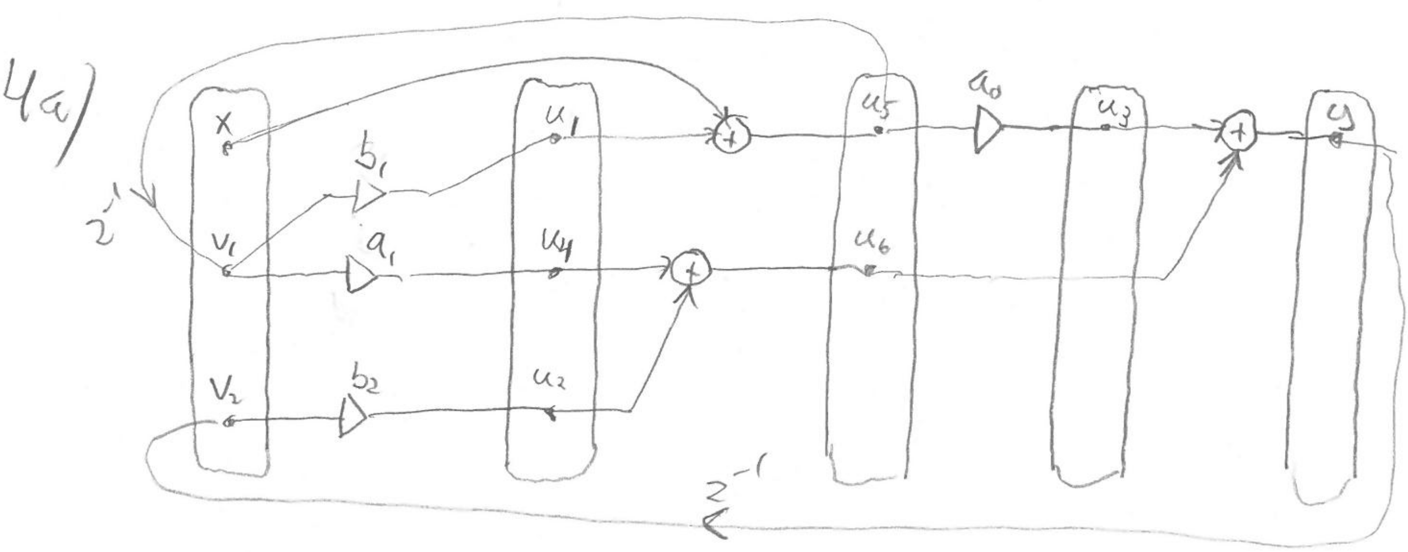
$\alpha_3 = \alpha_1$ (symmetry) $= 0{,}06898$

IV same as II

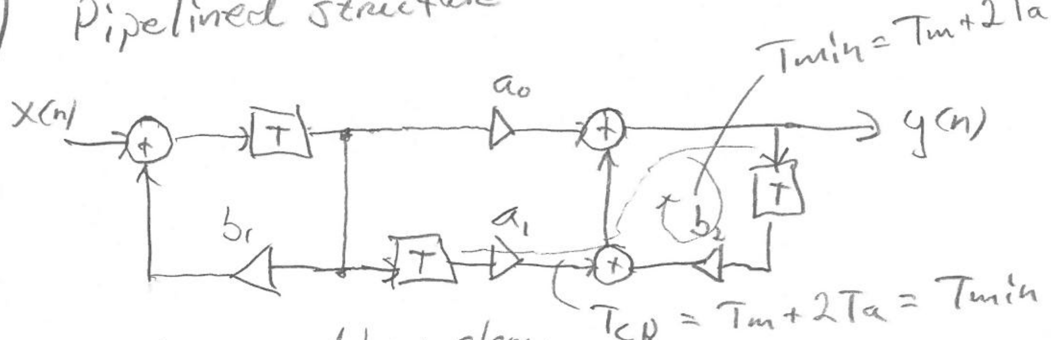
V same as I

due to symmetry

4a)



b) Pipelined structure



Propagate one delay elem. into the structure and

5a) Two roundings ⇒ $P_e = \frac{Q^2}{12}\cdot 2\cdot\sum_{n=0}^{\infty} g^2(n)$, $g(n) = (-b)^n u(n)$

variance

$G(z) = \frac{1}{1+bz^{-1}}$

$$\Rightarrow P_e = \frac{Q^2}{12}\, 2\cdot\sum_{n=0}^{\infty} (b^2)^n = \frac{Q^2}{12}\cdot\frac{2}{1-b^2} =$$

$$= \frac{2^{-20}}{6}\cdot\frac{1}{1-b^2} = 1.589\cdot 10^{-7}\cdot\frac{1}{1-b^2}$$

b) Branch 2 allpass $h_{ap}(n)$ ⇒ Inputs to both mult scaled

Output: $\sum_{n=0}^{\infty} h^2(n) = (1+h_{ap}(0))^2 + \underbrace{\sum_{n=1}^{\infty} h_{ap}^2(n)}_{=1-h_{ap}^2(0)} =$

since $\sum_{n=0}^{\infty} h_{ap}^2(n) = 1$

/ $h_{ap}(0) = b$

$$= (1+b)^2 + 1 - b^2$$

$$= 1 + 2b + b^2 + 1 - b^2 = 2(1-b)$$

$c = \frac{1}{\sqrt{2(1-b)}}$

6a) $H(z) = H_1(z^{L_2}) H_2(z)$

b) Transition bands of H_1, H_2: $\Delta_1 = 2\Delta L_2$, $\Delta_2 = 2\pi\left(\frac{1}{L_2} - \frac{1}{L}\right)$

Complexity - mult per output sample:

$$C = \frac{K}{\Delta_1 \cdot L} + \frac{K}{\Delta_2 \cdot L_2} = \frac{K}{2\Delta L_2 L} + \frac{K}{2\pi\left(1 - \frac{L_2}{L}\right)}$$

$$\frac{\partial C}{\partial L_2} = 0 \quad \text{for } L_2 = -\frac{a}{2} \underset{(-)}{+} \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

$$\text{where } a = -\frac{4\pi}{\frac{2\pi}{L} - 2\Delta L}, \quad b = \frac{2\pi L}{\frac{2\pi}{L} - 2\Delta L}$$

$$L = 24 \Rightarrow L_2^{(opt)} \approx 9.41$$

c) Select $L_1 = 2$, $L_2 = 12$, $L = L_1 L_2 = 24$

7a) set $\frac{sT}{2} = a + jb \Rightarrow |z| = \left|\frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}\right| = \left|\frac{1 + a + jb}{1 - a - jb}\right| = \frac{\sqrt{(1+a)^2 + b^2}}{\sqrt{(1-a)^2 + b^2}}$

$a < 0 \Leftrightarrow |z| < 1$ and $Re\left(\frac{sT}{2}\right) = Re(s)$ ($\frac{T}{2}$ positive)

$a > 0 \Leftrightarrow |z| > 1$ $\qquad Im = Im$

$a = 0 \Leftrightarrow |z| = 1$

$\therefore$ $Re(s) < 0 \Leftrightarrow |z| < 1$ stab. req. for poles of dig. filter

stab. req. for poles of anal filt.

b) $$s = \frac{2}{T}\frac{z-1}{z+1} = \Big/ z = e^{j\omega} \Big/ = \frac{2}{T}\frac{e^{j\omega T} - 1}{e^{j\omega T} + 1} = \frac{2}{T}\frac{e^{j\omega T/2}}{e^{j\omega T/2}}\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{e^{j\omega T/2} + e^{-j\omega T/2}}$$

$$= \frac{2}{T} \cdot \frac{2j\sin\left(\frac{\omega T}{2}\right)}{2\cos\left(\frac{\omega T}{2}\right)} = j \cdot \underbrace{\frac{2}{T}\tan\left(\frac{\omega T}{2}\right)}_{\omega_a}$$

c) $$\tau_g(\omega T) = -\frac{\partial \Phi(\omega T)}{\partial \omega} = \underbrace{-\frac{\partial \phi_a(\omega_a)}{\partial \omega_a}}_{\tau_{ga}(\omega)} \cdot \underbrace{\frac{\partial \omega_a}{\partial \omega}}_{= \frac{1}{\cos^2\left(\frac{\omega T}{2}\right)}} = \text{not const. if } \tau_{ga}(\omega) \text{ is const.}$$

due to