Solutions to the exam TSTE06 Digital Filters, 2015-06-02

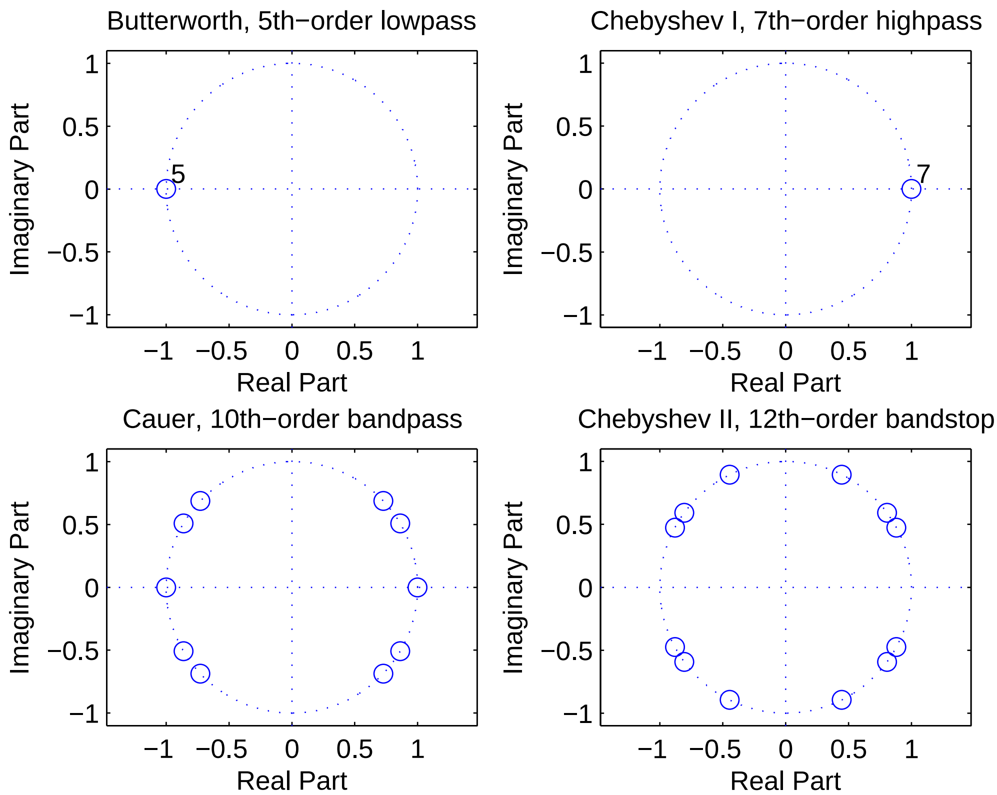
1 a) Lowpass, because H(1) = 4, H(-1)=0

b) In two's-complementary arithmetics, temporary overflows are okay in partial sums if the final sum is within the number range => only inputs to noninteger multipliers and the output need to be scaled. Safe scaling: never overflow. Lp-norm scaling: overflow with a certain probability.

c) 1) Inserted unit elements between two-ports, propagated into the filter using Kuroda's identities. 2) Directly interconnected adaptors with reflection-free ports.

d) The group delay is 10T seconds (10 samples).

e)



2) Band edges: $\omega_c T_{sample} = 0.5\ \pi$, $\omega_s T_{sample} = 0.84559\ \pi$.

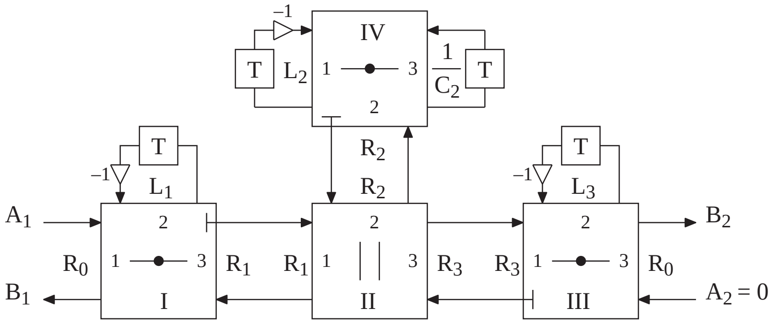
Bilinear transform $\Rightarrow \omega_{ac} = 1$, $\omega_{as} = 4.0417$.
Requirements on ω_{ac}, ω_{as}, and A_{min} => $15 \le \Theta \le 15$.
Normalized element values

$L'_1 = L'_3 = 0.9944$, $L'_2 = 0.0463$, $C'_2 = 1.0941$

Denormalize with $R_0 = 1$, $\omega_0 = \omega_{ac} = 1$ =>

$L_1 = L_3 = 0.9944,\ L_2 = 0.0463,\ C_2 = 1.0941$

Wave flow graph



Adaptor coefficients:

I:
$$R_1 = L_1 + R_0 = 1.9944$$
$$\alpha_1 = \frac{2R_0}{R_0 + L_1 + R_1} = 0.5014$$
$$\alpha_2 = 1 - \alpha_1 = 0.4986$$
$$\alpha_3 = 1$$

, III:
$$R_3 = L_3 + R_0 = 1.9944$$
$$\alpha_1 = 1$$
$$\alpha_2 = \frac{2L_3}{R_3 + L_3 + R_0} = 0.4986$$
$$\alpha_3 = 1 - \alpha_2 = 0.5014$$

IV:
$$R_2 = L_2 + \frac{1}{C_2} = 0.9603$$
$$\alpha_1 = \frac{2L_2}{L_2 + R_2 + \frac{1}{C_2}} = 0.039618$$
$$\alpha_2 = 1$$
$$\alpha_3 = 1 - \alpha_1 = 0.60382$$

, II:
$$\alpha_1 = \frac{\frac{2}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = 0.49057$$
$$\alpha_2 = \frac{\frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = 1.0189$$
$$\alpha_3 = 1 - \alpha_1 - \alpha_2 = 0.49057$$

3) Requirements for analog HP filter:

$$\omega_{ac} = \frac{2}{T}\tan\left(\frac{\omega_c T}{2}\right) = \frac{2}{T}\tan(0{,}325\pi) \approx \frac{2}{T}(1,631852)$$
$$\omega_{as} = \frac{2}{T}\tan\left(\frac{\omega_s T}{2}\right) = \frac{2}{T}\tan(0{,}2\pi) = \frac{2}{T}(0,726542)$$

Requirements for analog LP prototype filter:

$$\Omega_c = \frac{\omega_I^2}{\omega_{ac}} \qquad \Omega_s = \frac{\omega_I^2}{\omega_{as}}$$

Choose $\Omega_c = 1$ =>

$$\omega_I^2 = \omega_{ac} \qquad \Omega_s = \frac{\omega_{ac}}{\omega_{as}} \approx 2{,}24605$$

Order: nomogram => $N = 4$

Normalized poles:

$$S'_{1,2} = -0{,}1395360 \pm j0{,}9833792$$

$$S'_{3,4} = -0{,}3368697 \pm j0.4073290$$

Zeros: 4 at S = infinity

Denormalize with $\Omega_c = 1$ => $S_i = S'_i$

LP -> HP: $S = \frac{\omega_I^2}{s}$

Poles:

$$s_{1,2} = \frac{2}{T}(-0.2308 \pm j1.6267)$$

$$s_{3,4} = \frac{2}{T}(-1.96752 \pm j2.37905)$$

Zeros: 4 at s = 0

Transform to digital filter

$$z = \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$$

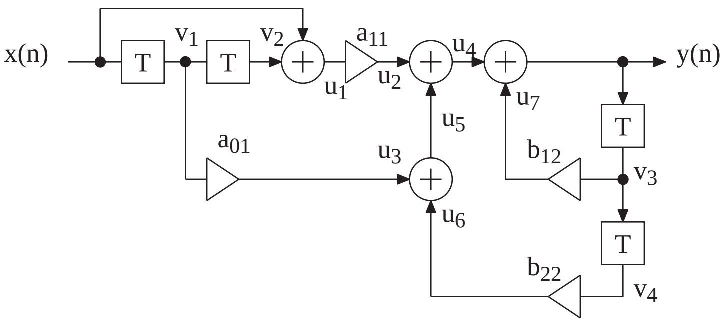
Poles:

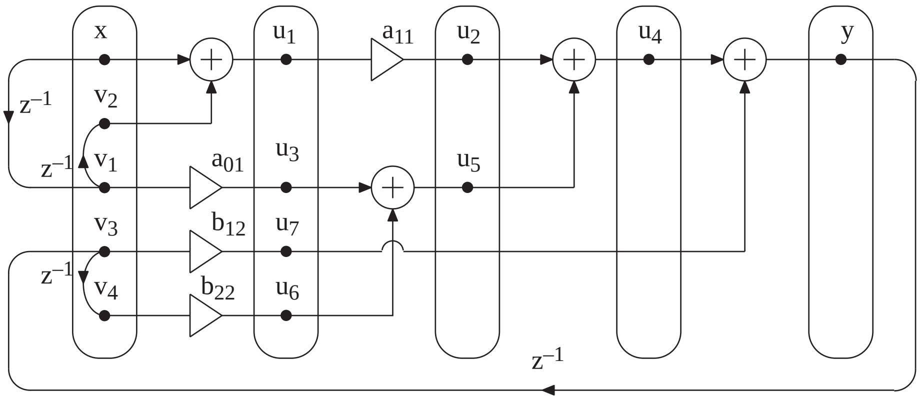
$$z_{1,2} = -0.4084 \pm j0.781869$$

$$z_{3,4} = -0.589725 \pm j0.328915$$

Zeros: 4 at z = 1.

4 a)





b)

$$u_1 = v_2 + x$$
$$u_3 = a_{01} v_1$$
$$u_7 = b_{12} v_3$$
$$u_6 = b_{22} v_4$$
$$u_2 = a_{11} u_1$$
$$u_5 = u_3 + u_6$$
$$u_4 = u_2 + u_5$$

$$u_7 = u_4 + u_5$$
$$y = u_4 + u_7$$
$$v_2 = v_1$$
$$v_1 = x$$
$$v_4 = v_3$$
$$v_3 = y$$

c) $T_{min} = T_{mult} + T_{add} = 1.1\ \mu s$, $T_{CP} = T_{mult} + 3T_{add} = 1.3\ \mu s$

5 a) $\sigma^2 = 2\frac{Q^2}{12}\sum_{n=0}^{\infty} h_1^2(n)$

$$H_1(z) = \frac{1}{1 - b_1 z^{-1}} \Leftrightarrow h_1(n) = b_1^n u(n)$$

where $b_1 = 0.875$

$$\sum_{n=0}^{\infty} h_1^2(n) = \sum_{n=0}^{\infty} b_1^{2n} = \frac{1}{1 - b_1^2}$$

$$\sigma^2 = 2\frac{Q^2}{12}\frac{1}{1-b_1^2} = 2\frac{2^{-28}}{12}\frac{1}{1-0{,}875^2} \approx 2{,}65 \cdot 10^{-9}$$

b) y(n) scaled by multiplying a_0 and a_2 by c.

$$c = \frac{1}{\sqrt{\sum_{n=0}^{\infty} h^2(n)}}$$

$$H(z) = \frac{a_0 + a_2 z^{-2}}{1 - b_1 z^{-1}} \Leftrightarrow h(n) = a_0 b_1^n u(n) + a_2 b_1^{n-2} u(n-2)$$

where $a_0 = 1$, $a_2 = -0.25$, $b_1 = 0.875$

$$h(n) = \begin{cases} 0, \; n < 0 \\ a_0, \; n = 0 \\ a_0 b_1, \; n = 1 \\ \left(a_0 + \frac{a_2}{b_1^2}\right) b_1^n, \; n > 1 \end{cases}$$

$$\sum_{n=0}^{\infty} h^2(n) = a_0^2 + (a_0 b_1)^2 + \left(a_0 + \frac{a_2}{b_1^2}\right)^2 \sum_{n=2}^{\infty} b_1^{2n}$$

$$= a_0^2 + (a_0 b_1)^2 + \left(a_0 + \frac{a_2}{b_1^2}\right)^2 \frac{b_1^4}{1 - b_1^2} = 2{,}9$$

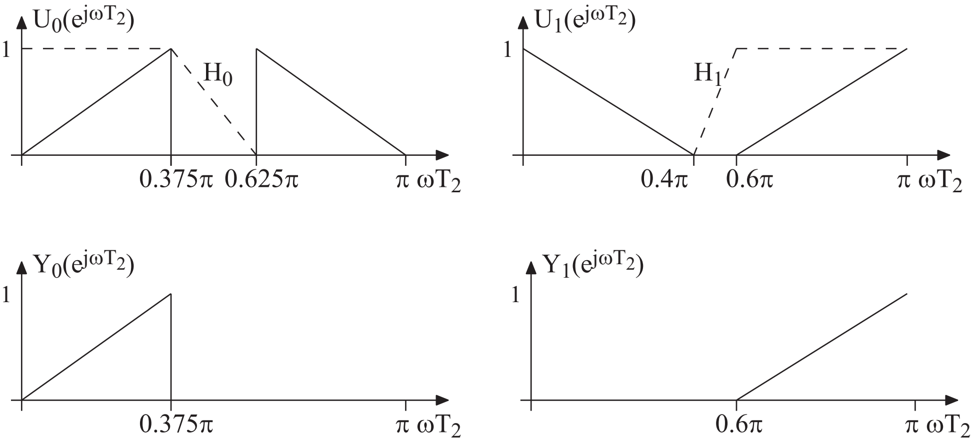
6 a) The passband of the lowpass filter G(z) is centred around $2\pi k$, k integer. Replacing z with z^M means that $2\pi k/M$ must be centred around $\pi/2$ for an overall highpass filter. Hence, $M = 4k$.

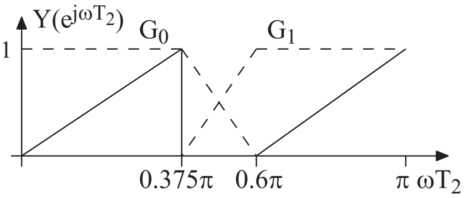
b) $\omega_c^G T = M(\omega_{c2}^H T - \pi/2)$, $\quad \omega_s^G T = M(\omega_{s2}^H T - \pi/2)$

$\omega_{c1}^F T = \omega_{c1}^H T$, $\quad \omega_{c2}^F T = \omega_{c2}^H T$

$\omega_{s1}^F T = \omega_{s2}^H T - 2\pi/M$, $\quad \omega_s^F T = \pi + 2\pi/M - \omega_s^H T$

7)





$V_k(e^{j\omega T_2}) = Y_k(e^{j\omega T_2})$

$W_k(e^{j\omega T_1}) = X_k(e^{j\omega T_1})$