$\boxed{5.2}$ The mean SNR is $\gamma_0 = 13\,dB \approx 20$

a) The BER for coherent BPSK over a Rayleigh fading channel is

$$P_b = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}}\right) \approx 0{,}012$$

b) According to (5.39) the BER for coherent BPSK with M branch maximal ratio combining is

$$P_b = \left(\frac{1-\mu}{2}\right)^M \sum_{i=0}^{M-1} \binom{M-1+i}{i} \left(\frac{1+\mu}{2}\right)^i,$$

where $\mu = \sqrt{\frac{\gamma_0}{\gamma_0+1}} \approx 0{,}9759$.
For $M = 2, 3, 4$ we obtain $P_b = 4{,}39 \cdot 10^{-4};\ 1{,}73 \cdot 10^{-5};\ 7{,}23 \cdot 10^{-7}$, respectively. Thus we must have at least 4 diversity branches.

c) The BER for coherent BPSK over an AWGN channel of SNR $\gamma = 13\,dB$ is

$$P_b = Q\left(\sqrt{2\gamma}\right) \approx Q(6{,}32) \approx 1{,}3 \cdot 10^{-10}$$

5.3 The BER for non-coherent FSK over Rayleigh fading channel is

$$P_b = \frac{1}{2+\gamma_0} = 10^{-4} \Rightarrow \gamma_0 = 10^4 - 2 = 9998 \approx 40\,dB$$

a) In space diversity the receiver uses several antennas to get the different branches. Thus the transmitter sends withe the same energy and the SNR on each branch is γ_0. The BER for 2 branch selection combining and non-coherent FSK modulation is (5.51)

$$P_b = \frac{1}{2}\left(\sum_{k=0}^{2}(-1)^k\binom{2}{k}\frac{\gamma_0}{2k+\gamma_0}\right) = \frac{1}{2}\left(1 - \frac{2\gamma_0}{2+\gamma_0} + \frac{\gamma_0}{4+\gamma_0}\right) =$$

$$= \frac{\gamma_0^2+6\gamma_0+8-8\gamma_0-2\gamma_0^2+2\gamma_0+\gamma_0^2}{2(2+\gamma_0)(4+\gamma_0)} = \frac{4}{(2+\gamma_0)(4+\gamma_0)}$$

Setting $P_b = 10^{-4}$ we obtain $\gamma_0^2+6\gamma_0-39992=0 \Rightarrow$
$\gamma_0 \approx 197 \approx 23\,dB$. Thus the gain is 17 dB.

b) In frequency diversity the signal is sent on two different frequency bands. Thus the transmitted energy has to be divided on those two frequencies. The received SNR on the two branches is $\gamma_0/2$.
As in a) we obtain

$$P_b = \frac{4}{(2+\gamma_0/2)(4+\gamma_0/2)} = \frac{16}{(4+\gamma_0)(8+\gamma_0)}$$

Setting $P_b = 10^{-4}$ we get $\gamma_0^2+12\gamma_0-159968=0 \Rightarrow$
$\gamma_0 \approx 394 \approx 26\,dB$. Thus the gain is 14 dB.

$\boxed{5.8}$ The probability density function for the SNR γ in a flat Rayleigh fading is

$$p_\gamma(\gamma) = \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \quad , \gamma \geq 0 \quad \text{and} \quad 0, \text{ otherwise.}$$

a) The probability that the SNR γ is below 15dB is

$$P_\gamma(10^{1,5}) = \int_0^{10^{1,5}} \frac{1}{\gamma_0} e^{-\frac{\gamma}{\gamma_0}} d\gamma = \left[-e^{-\frac{\gamma}{\gamma_0}}\right]_0^{10^{1,5}} = 1 - e^{-\frac{10^{1,5}}{\gamma_0}} \leq 0{,}1$$

this gives $-\frac{10^{1,5}}{\gamma_0} \geq \ln 0{,}9 \Rightarrow \gamma_0 \geq -\frac{10^{1,5}}{\ln 0{,}9} \approx 300{,}14$,
which is $\gamma_0 \gtrsim 24{,}77$ dB

b) The distribution function with M branch selection combining is

$$P_\gamma(\gamma) = \left(1 - e^{-\frac{\gamma}{\gamma_0}}\right)^M.$$

The requirement is now expressed as

$$P_\gamma(10^{1,5}) = \left(1 - e^{-\frac{10^{1,5}}{\gamma_0}}\right)^2 \leq 0{,}1 \Rightarrow e^{-\frac{10^{1,5}}{\gamma_0}} \geq 1 - \sqrt{0{,}1} \Rightarrow$$

$$\Rightarrow -\frac{10^{1,5}}{\gamma_0} \geq \ln\left(1 - \sqrt{0{,}1}\right) \Rightarrow \gamma_0 \geq -\frac{10^{1,5}}{\ln\left(1-\sqrt{0{,}1}\right)} \approx 83{,}189,$$

whish is $\gamma \gtrsim 19{,}2$ dB

$\boxed{5.9}$ As in problem 5.8 we obtain the mean SNR γ_0 as

$$P_r(10^{0.5}) = 1 - e^{-\frac{10^{0.5}}{\gamma_0}} = 0{,}05 \Rightarrow \gamma_0 = -\frac{10^{0.5}}{\ln 0{,}95} \approx 61{,}65$$

For switched combining the pdf is given by (see 5.52)

$$p_r(\gamma) = \begin{cases} (1+q_x)\frac{1}{\gamma_0}e^{-\frac{\gamma}{\gamma_0}}, & \gamma \geq \gamma_x \\ q_x \frac{1}{\gamma_0} e^{-\frac{\gamma}{\gamma_0}}, & \gamma < \gamma_x \end{cases}$$

where $q_x = 1 - e^{-\frac{\gamma_x}{\gamma_0}}$, and γ_x is the threshold SNR.

The distribution function for the SNR γ is

$$P_r(\gamma) = \begin{cases} q_x\left(1 - e^{-\frac{\gamma}{\gamma_0}}\right), & \gamma < \gamma_x \\ (1+q_x)\left(1 - e^{-\frac{\gamma}{\gamma_0}}\right) - q_x, & \gamma \geq \gamma_x \end{cases}$$

We have the requirement $P_r(10^{0,5}) \leq 0{,}01$

If $\gamma_x \leq 5\,dB$, then

$$P_r(10^{0,5}) = (1+q_x)\left(1 - e^{-\frac{10^{0,5}}{\gamma_0}}\right) - q_x = (1+q_x)\left(1 - e^{-\frac{10^{0,5}}{(-10^{0,5}/\ln 0{,}95)}}\right) - q_x =$$

$$= (1+q_x)\,0{,}05 - q_x = 0{,}05 - 0{,}95 q_x \leq 0{,}01 \Rightarrow q_x \geq \frac{0{,}04}{0{,}95}$$

Thus $1 - e^{-\frac{\gamma_x}{\gamma_0}} \geq \frac{4}{95} \Rightarrow e^{-\frac{\gamma_x}{\gamma_0}} \leq \frac{91}{95} \Rightarrow -\frac{\gamma_x}{\gamma_0} \leq \ln\frac{91}{95}$

$\Rightarrow \gamma_x \geq -\gamma_0 \ln(91/95) \approx 2{,}652 \Rightarrow \gamma_x \gtrsim 10\log_{10}(2{,}652) \approx 4{,}236\,dB$

In this case we obtain $4{,}236\,dB \leq \gamma_x \leq 5\,dB$

If $\gamma_x \geq 5\,dB$, then

$$P_r(10^{0,5}) = q_x\left(1 - e^{-\frac{\gamma}{\gamma_0}}\right) = q_x\left(1 - e^{-\frac{10^{0,5}}{\gamma_0}}\right) = 0{,}05 q_x \leq 0{,}01 \Rightarrow q_x \leq 0{,}2.$$

Thus

$$1 - e^{-\frac{\gamma_x}{\gamma_0}} \leq 0{,}2 \Rightarrow e^{-\frac{\gamma_x}{\gamma_0}} \geq 0{,}8 \Rightarrow -\frac{\gamma_x}{\gamma_0} \geq \ln 0{,}8 \Rightarrow$$

$$\gamma_x \leq -\gamma_0 \ln 0{,}8 \approx 13{,}757 \Rightarrow \gamma_x \lesssim 10\log_{10}(13{,}757) \approx 11{,}385\,dB$$

In this case we obtain $5\,dB \leq \gamma_x \lesssim 11{,}385\,dB$

Totally $4{,}236\,dB \leq \gamma_x \leq 11{,}385\,dB$

$\boxed{5.11}$ We use a selection combining where the M different branches have average SNR $\gamma_0' = \frac{\gamma_0}{M}$.

a) The distribution function of the SNR γ of the selection combined signal is

$$P_\Gamma(\gamma) = \prod_{i=1}^{M} P_{\Gamma_i}(\gamma) = \left(1 - e^{-\frac{\gamma}{\gamma_0'}}\right)^M.$$

The requirement for time availability can now be written as

$$P_\Gamma(10^{1,0}) = \left(1 - e^{-\frac{10}{\gamma_0/M}}\right)^M \leq 0{,}02 \Rightarrow \left(1 - e^{-\frac{M}{10}}\right)^M \leq 0{,}02,$$

using that $\gamma_0 = 20\,dB \approx 100$.

The left hand side of the inequality is $0{,}032\ldots$ for $M=2$ and $0{,}017\ldots$ for $M=3$. Thus the minimum number of branches needed is $M=3$.

b) The function $\left(1 - e^{-\frac{M}{10}}\right)^M$ achives its minimum at integer numbers in $M=7$. This minimum is $0{,}0081937$. Thus the highest achievable time-availability is $1 - 0{,}0081937$, which is approximately $99{,}2\,\%$.

This could be achived with $M=7$ branches.