$\boxed{4.15}$ The BER for non-coherent FSK in AWGN with SNR γ is

$$P_b(\gamma) = \frac{1}{2} e^{-\frac{\gamma}{2}}$$

The BER over a fading channel with SNR following the distribution function $P(\gamma)$ is

$$P_b = E\{P_b(\gamma)\} = \int_{-\infty}^{\infty} P_b(\gamma)\, dP(\gamma)$$

In this special case we have:

$$P_b(\gamma_0) = \int_0^{\infty} \frac{1}{2} e^{-\frac{\gamma}{2}}\, d\left(1 - e^{-\frac{\gamma}{\gamma_0}} - \frac{\gamma}{\gamma_0} e^{-\frac{\gamma}{\gamma_0}}\right) =$$

$$= \frac{1}{2}\int_0^{\infty} e^{-\frac{\gamma}{2}} \left(e^{-\frac{\gamma}{\gamma_0}} \cdot \frac{1}{\gamma_0} - \frac{1}{\gamma_0} e^{-\frac{\gamma}{\gamma_0}} + \frac{\gamma}{\gamma_0^2} e^{-\frac{\gamma}{\gamma_0}} \right) d\gamma =$$

$$= \frac{1}{2\gamma_0^2} \int_0^{\infty} \gamma e^{-\gamma \frac{2+\gamma_0}{2\gamma_0}}\, d\gamma = \frac{1}{2\gamma_0^2} \cdot \left(-\frac{2\gamma_0}{2+\gamma_0}\right) \int_0^{\infty} \gamma\, d e^{-\gamma \frac{2+\gamma_0}{2\gamma_0}} =$$

$$= -\frac{1}{\gamma_0(2+\gamma_0)} \left(\gamma e^{-\gamma \frac{2+\gamma_0}{2\gamma_0}} \Big|_0^{\infty} - \int_0^{\infty} e^{-\gamma \frac{2+\gamma_0}{2\gamma_0}}\, d\gamma \right) =$$

$$= \frac{1}{\gamma_0(2+\gamma_0)} \cdot \frac{-2\gamma_0}{2+\gamma_0} e^{-\gamma \frac{2+\gamma_0}{2\gamma_0}} \Big|_0^{\infty} = \frac{2}{(2+\gamma_0)^2}$$

Putting $\gamma_0 = 18\,dB \approx 63{,}1$ we obtain

$$P_b(63{,}1) \approx \frac{2}{(2+63{,}1)^2} \approx 4{,}72 \cdot 10^{-4}$$

$\boxed{4.16}$ The signal data rate is $R_b = 9600$ bit/s. This gives the symbol (bit) time duration $T_b = T_S = \frac{1}{9600}$ s.

a) The required maximum out-of-band power is 1% which corresponds to -20 dB ($10 \cdot \log_{10}(0{,}01) = -20$). From figure 4.14 (page 223) we can read out that it is achieved for $B_{RF} \cdot T_S \geq 10$.

Thus $B_{RF} \geq \frac{10}{T_S} = 10 \cdot R_b = 96$ kHz.

b) We can assume that the two signal frequencies are 0 and $\frac{1}{T_S}$. Then in equation (4.108 on page 241 we have

$S_0(f) = \sqrt{E_S T_S}\,\mathrm{sinc}(fT_S)\,e^{-j\pi f T_S}$ and $S_1(f) = -\sqrt{E_S T_S}\,\mathrm{sinc}(fT_S - 1)\,e^{-j\pi f T_S}$.

Substituting these expressions in 4.108 we get the power density spectrum:

$$S_e(f) = \frac{E_S}{4}\left(\mathrm{sinc}(fT_S) + \mathrm{sinc}(fT_S - 1)\right)^2$$

The center frequency of the baseband modulated signal is $\frac{fT_S}{2}$. The out-of-band power can be calculated as

$$\frac{\int_{\frac{(fT_S + B_{RF})}{2}}^{\infty} S_e(f)\,df}{\int_{\frac{fT_S}{2}}^{\infty} S_e(f)\,df}$$

Putting this to be 1% we could approximately get $B_{RF} \approx \frac{8}{T_S} = 8 \cdot 9600 = 76{,}8$ kHz.

4.18 The fading channel's impulse response is

$h(t) = \mathcal{F}^{-1}\{H(f)\} = \delta(t) - \rho\,\delta(t-\tau)$

This means that the received signal is

$r(t) = s(t) - \rho\, s(t-\tau) + n(t)$, where $n(t)$ is the AWGN noise.

The optimal correlation detector calculates

$$\hat{r} = \int_0^T \sqrt{\tfrac{2}{T}} \cos\left(\tfrac{2\pi}{T}t\right) \cdot r(t)\, dt$$

and decides upon the sign of $\hat{r}$ to be the transmitted value of a_k.

a) $\rho = 0$, $\tau = 0$ $\Rightarrow$

$$\hat{r} = \int_0^T \sqrt{\tfrac{2}{T}} \cos\left(\tfrac{2\pi}{T}t\right)\left(a_k\sqrt{\tfrac{2E_b}{T}}\cos\left(\tfrac{2\pi}{T}t\right) + n(t)\right)dt =$$

$= a_k\sqrt{E_b} + n_0$, where n_0 is a stochastic variable which is normally distributed with zero mean and variance $\frac{N_0}{2}$. Now the BER is $P_b = P_r\{n_0 \geq \sqrt{E_b}\} =$

$= P_r\left\{\frac{n_0}{\sqrt{N_0/2}} \geq \sqrt{\frac{2E_b}{N_0}}\right\} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

For $E_b/N_0 = 10\,dB = 10$ we have $P_b = Q(\sqrt{20}) \approx 3.9 \times 10^{-6}$

$\boxed{4.18}$ b) $\rho = \frac{1}{2}, \ \tau = \frac{T}{8} \Rightarrow$

$$\hat{r} = \int_0^T \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi}{T} t\right) \left(s(t) - \frac{1}{2} s\left(t - \frac{T}{8}\right) + n(t) \right) dt =$$

$$= a_k \sqrt{E_b} + n_0 - \frac{\sqrt{E_b}}{T} a_{k-1} \int_0^{T/8} \cos\left(\frac{2\pi}{T} t\right) \cos\left(\frac{2\pi}{T} t - \frac{\pi}{4}\right) dt -$$

$$- \frac{\sqrt{E_b}}{T} a_k \int_{T/8}^{T} \cos\left(\frac{2\pi}{T} t\right) \cos\left(\frac{2\pi}{T} t - \frac{\pi}{4}\right) dt =$$

$$= a_k \sqrt{E_b} + n_0 - \frac{\sqrt{E_b}}{2T} a_{k-1} \int_0^{T/8} \left(\cos\left(\frac{4\pi}{T} t - \frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \right) dt -$$

$$- \frac{\sqrt{E_b}}{2T} a_k \int_{T/8}^{T} \left(\cos\left(\frac{4\pi}{T} t - \frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \right) dt =$$

$$= a_k \sqrt{E_b} + n_0 - \frac{\sqrt{E_b}}{2T} a_{k-1} \left(\frac{T}{4\pi} \sin\left(\frac{4\pi}{T} t - \frac{\pi}{4}\right) \Big|_0^{T/8} + \frac{\sqrt{2}}{2} \cdot \frac{T}{8} \right) -$$

$$- \frac{\sqrt{E_b}}{2T} a_k \left(\frac{T}{4\pi} \sin\left(\frac{4\pi}{T} t - \frac{\pi}{4}\right) \Big|_{T/8}^{T} + \frac{\sqrt{2}}{2} \cdot \frac{7T}{8} \right) =$$

$$= a_k \sqrt{E_b} + n_0 - \left(\frac{\sqrt{2}}{32} + \frac{\sqrt{2}}{8\pi} \right) a_{k-1} \sqrt{E_b} - \left(\frac{\sqrt{2}}{32} - \frac{\sqrt{2}}{8\pi} \right) a_k \sqrt{E_b} =$$

$$= \left(1 - \frac{7\sqrt{2}}{32} + \frac{\sqrt{2}}{8\pi} \right) a_k \sqrt{E_b} - \left(\frac{\sqrt{2}}{32} + \frac{\sqrt{2}}{8\pi} \right) a_{k-1} \sqrt{E_b} + n_0$$

Now we have two equally probable cases:

I: $a_{k-1} = a_k \Rightarrow \hat{r} = a_k \left(1 - \frac{\sqrt{2}}{4} \right) \sqrt{E_b} + n_0 \approx a_k \sqrt{0{,}418 E_b} + n_0$

II: $a_{k-1} = -a_k \Rightarrow \hat{r} = a_k \left(1 - \frac{3\sqrt{2}}{16} + \frac{\sqrt{2}}{4\pi} \right) \sqrt{E_b} + n_0 \approx a_k \sqrt{0{,}718 E_b} + n_0$

Now the BER is $P_b = \frac{1}{2} \left(P_r\{ n_0 \geq \sqrt{0{,}418 E_b} \} + P_r\{ n_0 \geq \sqrt{0{,}718 E_b} \} \right) =$

$$= \frac{1}{2} \left(P_r\left\{ \frac{n_0}{\sqrt{N_0/2}} \geq \sqrt{0{,}836 E_b/N_0} \right\} + P_r\left\{ \frac{n_0}{\sqrt{N_0/2}} \geq \sqrt{1{,}436 E_b/N_0} \right\} \right) =$$

$$= \frac{1}{2} \left(Q(\sqrt{8{,}36}) + Q(\sqrt{14{,}36}) \right) \approx 9{,}7 \times 10^{-4}$$

The first case gives the dominant term in the calculation of P_b. Thus we need to have a new $\hat{E}_b$ such that $Q\left(\sqrt{0{,}836 \hat{E}_b/N_0}\right) = 2Q(\sqrt{20})$, which is an increase of $10 \log\left(\frac{1}{0{,}418}\right) \approx 3{,}79$ dB

$\boxed{14.18}$ c) $\rho = \frac{1}{2} \quad \tau = \frac{T}{4} \Rightarrow$

$$\hat{r} = \int_0^T \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi}{T} t\right) \left(s(t) - \frac{1}{2} s\left(t - \frac{T}{4}\right) + n(t) \right) dt = \dots$$

$$= a_k \sqrt{E_b} + n_0 - \frac{\sqrt{E_b}}{2T} a_{k-1} \left(\frac{T}{4\pi} \sin\left(\frac{4\pi}{T} t - \frac{\pi}{2}\right) \Big|_0^{T/4} + \cos\left(\frac{\pi}{2}\right) \frac{T}{4} \right) -$$

$$- \frac{\sqrt{E_b}}{2T} a_k \left(\frac{T}{4\pi} \sin\left(\frac{4\pi}{T} t - \frac{\pi}{2}\right) \Big|_{T/4}^{T} + \cos\left(\frac{\pi}{2}\right) \frac{3T}{4} \right) =$$

$$= \left(1 + \frac{1}{4\pi}\right) a_k \sqrt{E_b} - \frac{1}{4\pi} a_{k-1} \sqrt{E_b} + n_0$$

The two cases are now:

$\mathrm{I}: a_{k-1} = a_k \Rightarrow \hat{r} = a_k \sqrt{E_b} + n_0$

$\mathrm{II}: a_{k-1} = a_k \Rightarrow \hat{r} = a_k \left(1 + \frac{1}{2\pi}\right) \sqrt{E_b} + n_0 \approx a_k \sqrt{1.344 E_b} + n_0$

The BER is

$$P_b = \frac{1}{2} \left(\Pr\{ n_0 \geq \sqrt{E_b} \} + \Pr\{ n_0 \geq \sqrt{1{,}344 E_b} \} \right) =$$

$$= \frac{1}{2} \left(\Pr\left\{ \frac{n_0}{\sqrt{N_0/2}} \geq \sqrt{\frac{2E_b}{N_0}} \right\} + \Pr\left\{ \frac{n_0}{\sqrt{N_0/2}} \geq \sqrt{\frac{2{,}688 E_b}{N_0}} \right\} \right) =$$

$$= \frac{1}{2} \left(Q(\sqrt{20}) + Q(\sqrt{26{,}88}) \right) \approx 2 \cdot 10^{-6}$$

The new value of the bit energy $\hat{E}_b$ has to satisfy

$$Q(\sqrt{20}) = \frac{1}{2} \left(Q \sqrt{2\hat{E}_b / N_0} + Q\left(\sqrt{2{,}688 \hat{E}_b / N_0}\right) \right)$$

Numerically we can calculate that

$\hat{E}_b / N_0 \cong 9{,}33 \approx 9{,}7\,dB$ which is a decrease of $0{,}3\,dB$.

$\boxed{4.20}$ The coherence time of the channel $T_d \approx 0{,}1\,s$ is much bigger than the symbol (bit) duration $T_S = \frac{1}{R_s} = 1\,ms = 0{,}001\,s$. Thus we have a slowly fading conditions. The coherence bandwidth of the channel is $B_m \approx \frac{1}{T_m} = 10\,kHz$. The transmission bandwidth for BFSK is approximately $B_{RF} \approx 2 R_b = 2\,kHz$. Since $B_m >> B_{RF}$ we have flat fading channel. For the BER of slowly, flat fading channel we have

$$P_b = \frac{1}{2+\Gamma_0} \leq 10^{-3} \Rightarrow \Gamma_0 \geq 998 \approx 30\,dB$$

4.23 The BER for DPSK modulation in AWGN channel of SNR γ is given by

$$P_b(\gamma) = \frac{1}{2} e^{-\gamma}$$

The average BER is

$$P_b = E[P_b(\gamma)] = \int_0^\infty \frac{1}{2} e^{-\gamma} \frac{\gamma}{\alpha^2} \exp\left(-\frac{\gamma}{\alpha}\right) d\gamma =$$

$$= \frac{1}{2\alpha^2} \int_0^\infty \gamma \cdot e^{-\gamma \frac{\alpha+1}{\alpha}} d\gamma = \frac{1}{2\alpha^2} \cdot \frac{-\alpha}{\alpha+1} \left(\gamma \cdot e^{-\gamma \frac{\alpha+1}{\alpha}} \Big|_0^\infty - \int_0^\infty e^{-\gamma \frac{\alpha+1}{\alpha}} d\gamma \right) =$$

$$= + \frac{1}{2\alpha(\alpha+1)} \cdot \frac{-\alpha}{\alpha+1} \cdot e^{-\gamma \frac{\alpha+1}{\alpha}} \Big|_0^\infty = \frac{1}{2(\alpha+1)^2}$$

The requirement is now

$$P_b \leq 10^{-4} \Rightarrow \frac{1}{2(\alpha+1)^2} \leq 10^{-4} \Rightarrow \alpha \geq \sqrt{5000} - 1 \approx 69{,}7$$

The mean-SNR γ_0 is calculated as

$$\gamma_0 = E[\gamma] = \int_{-\infty}^{\infty} p(\gamma) \cdot \gamma \, d\gamma = \int_0^\infty \frac{\gamma^2}{\alpha^2} e^{-\frac{\gamma}{\alpha}} d\gamma = \alpha \int_0^\infty \gamma^2 e^{-\gamma} d\gamma =$$

$$= \alpha \left(-\gamma^2 e^{-\gamma} \Big|_0^\infty + \int_0^\infty 2\gamma e^{-\gamma} d\gamma \right) = 2\alpha \left(-\gamma e^{-\gamma} \Big|_0^\infty + \int_0^\infty e^{-\gamma} d\gamma \right) =$$

$$= -2\alpha \, e^{-\gamma} \Big|_0^\infty = 2\alpha \geq 139{,}6 \approx 21{,}4 \, dB$$

4.25 a) The ordinary correlation receiver of BPSK modulation estimates the bit value by calculating

$$\hat{r}_n = \sqrt{E_b} \cdot s_n + n_n \quad , \text{where } n_n \sim \mathcal{N}(0, \tfrac{N_0}{2})$$

The BER is given by the probability

$$P_b = Pr\{ n_n \geq \sqrt{E_b} \} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

For this non ideal channel we can have two situations:

I $s_{n-1} = -s_n \Rightarrow r_n = 1{,}4 s_n + n_n$ and

II $s_{n-1} = +s_n \Rightarrow r_n = 0{,}2 s_n + n_n$,

Which are equally probable.

Thus the BER for this channel is given by

$$P_b = \frac{1}{2}\left(Q\left(\frac{1{,}4}{\sqrt{0{,}1}}\right) + Q\left(\frac{0{,}2}{\sqrt{0{,}1}}\right)\right) \approx 0{,}132$$

The BER for the system without ISI is

$$P_b = Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right) = Q\left(\frac{1}{\sqrt{0{,}1}}\right) \approx 7{,}89 \cdot 10^{-4} \quad , \text{since } E_b = 1.$$

b) We have the estimation $\hat{s}_n = \sum_{j=-1}^{1} c_j r_{n-j}$

We have $h_{-1} = 0$, $h_0 = 0{,}8$, $h_1 = -0{,}6$, $h_2 = 0$. We calculate $x_0 = h_0^2 + h_1^2 = 1$, $x_1 = h_0 h_1 = -0{,}48$. Since $N_0/2 = 0{,}1$, the matrices Γ and Λ are

$$\Gamma = \begin{bmatrix} 1{,}1 & -0{,}48 & 0 \\ -0{,}48 & 1{,}1 & -0{,}48 \\ 0 & -0{,}48 & 1{,}1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} -0{,}6 \\ 0{,}8 \\ 0 \end{bmatrix}$$

We obtain the optimal vector C by

$$C_{opt} = \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \Gamma^{-1}\Lambda = \begin{bmatrix} 1{,}1887 & 0{,}6407 & 0{,}2796 \\ 0{,}6407 & 1{,}4682 & 0{,}6407 \\ 0{,}2796 & 0{,}6407 & 1{,}1887 \end{bmatrix} \begin{bmatrix} -0{,}6 \\ 0{,}8 \\ 0 \end{bmatrix} = \begin{bmatrix} -0{,}2007 \\ 0{,}7902 \\ 0{,}3448 \end{bmatrix}$$

4.25 c) The q coefficients are calculated as follows:

$$q_0 = \sum_{k=-1}^{1} c_k h_{-k} = (-0{,}2007)(-0{,}6) + 0{,}7902 \cdot 0{,}8 \approx 0{,}75258$$

$$q_1 = \sum_{k=-1}^{1} c_k h_{1-k} = 0{,}7902 \cdot (-0{,}6) + 0{,}3448 \cdot 0{,}8 \approx -0{,}19828$$

$$q_{-1} = \sum_{k=-1}^{1} c_k h_{-1-k} = -0{,}2007 \cdot 0{,}8 \approx -0{,}16056$$

$$q_2 = \sum_{k=-1}^{1} c_k h_{2-k} = 0{,}3448(-0{,}6) \approx -0{,}20888$$

For the variance of the the noise sample z_n we have

$$\sigma_z^2 = \sigma_n^2 \sum_{k=-1}^{1} c_k^2 = \frac{N_0}{2}\left(0{,}0403 + 0{,}6244 + 0{,}1189\right) = 0{,}0784$$

d) The output from the equalizer is

$$r_n' = 0{,}75258 s_n - 0{,}19828 s_{n-1} - 0{,}16056\, s_{n+1} - 0{,}20888 s_{n-2} + z_n$$

We have 8 different cases for the values of the bits s_{n-1}, s_{n+1} and s_{n+2} $(s_n = 1)$

s_{n-1}	s_{n+1}	s_{n-2}	$r_n' - z_n$
1	1	1	0,1868
1	1	-1	0,6006
1	-1	1	0,508
1	-1	-1	0,9218
-1	1	1	0,5834
-1	1	-1	0,9972
-1	-1	1	0,9046
-1	-1	-1	1,3184

These cases are equally probable and thus the BER is

$$P_b = \frac{1}{8}\left(Q\left(\frac{0{,}1868}{\sqrt{0{,}0784}}\right) + Q\left(\frac{0{,}6006}{\sqrt{0{,}0784}}\right) + Q\left(\frac{0{,}508}{\sqrt{0{,}0784}}\right) + Q\left(\frac{0{,}9218}{\sqrt{0{,}0784}}\right) + \right.$$

$$\left. + Q\left(\frac{0{,}5834}{\sqrt{0{,}0784}}\right) + Q\left(\frac{0{,}9972}{\sqrt{0{,}0784}}\right) + Q\left(\frac{0{,}9046}{\sqrt{0{,}0784}}\right) + Q\left(\frac{1{,}3184}{\sqrt{0{,}0784}}\right)\right) \approx$$

$\approx 0{,}0402$, compared to $0{,}132$ without the equalizer.