4.5 a) We use the maximum a posteriori (MAP) criterium, i.e. Decide for s_1 iff $P(s_1|r) \geq P(s_2|r)$. The a posteriori probabilities can be determined as follows:

$$P(s_i|r) = \frac{P(s_i, r)}{P(r)} = \frac{P(r|s_i).P(s_i)}{P(r)} = \frac{P(s_i)}{P(r)} \cdot P(r|s_i)$$

Since the a priori probabilities for both signals are equal, i.e. $P(s_1) = P(s_2) = \frac{1}{2}$, the above condition is transformed into: Decide for s_1 iff $P(r|s_1) \geq P(r|s_2)$. The MAP optimal decision region is determined by the inequality

$$P(r|s_1) = f(r|s_1) = \frac{e^{-\frac{r^2}{2\sigma_1^2}}}{\sqrt{2\pi\sigma_1^2}} \geq \frac{e^{-\frac{r^2}{2\sigma_2^2}}}{\sqrt{2\pi\sigma_2^2}} = f(r|s_2) = P(r|s_2)$$

$$\Rightarrow e^{\frac{r^2}{2}\cdot\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2\sigma_2^2}} \geq \frac{\sigma_1}{\sigma_2} \Rightarrow \frac{r^2}{2}\cdot\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2\sigma_2^2} \geq \ln\left(\frac{\sigma_1}{\sigma_2}\right)$$

$$\xRightarrow{(\sigma_1^2-\sigma_2^2>0)} |r| \geq \sigma_1\sigma_2\sqrt{\frac{2\ln\left(\frac{\sigma_1}{\sigma_2}\right)}{\sigma_1^2-\sigma_2^2}}$$ is the region where we choose s_1.

b) In the case of independent samples we have $f(r_1, r_2|s_i) = f(r_1|s_i).f(r_2|s_i)$

Similarly the decision region for s_1 is determined by

$$P(r_1, r_2|s_1) = f(r_1|s_1).f(r_2|s_1) = \frac{e^{-\frac{r_1^2+r_2^2}{2\sigma_1^2}}}{(\sqrt{2\pi\sigma_1^2})^2} \geq \frac{e^{-\frac{r_1^2+r_2^2}{2\sigma_2^2}}}{(\sqrt{2\pi\sigma_2^2})^2} = P(r_1, r_2|s_2)$$

$$\Rightarrow e^{\frac{r_1^2+r_2^2}{2}\cdot\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2\sigma_2^2}} \geq \left(\frac{\sigma_1}{\sigma_2}\right)^2 \Rightarrow \frac{r_1^2+r_2^2}{2}\cdot\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2\sigma_2^2} \geq 2\ln\left(\frac{\sigma_1}{\sigma_2}\right)$$

$$\xRightarrow{(\sigma_1^2-\sigma_2^2>0)} r_1^2+r_2^2 \geq \frac{4\sigma_1^2\sigma_2^2\ln\left(\frac{\sigma_1}{\sigma_2}\right)}{\sigma_1^2-\sigma_2^2}$$ is the region where we choose s_1.

4.8 Since we have uniform distribution over the interval $[0,a]$ of the amplitude X, the probability density function of X is $f_X(x) = \begin{cases} \frac{1}{a}, & \text{if } x \in [0,a] \\ 0, & \text{otherwise} \end{cases}$.

If we assume that the modulated signal $s(t)$ has bit energy E_b, then the "faded" signal $X.s(t)$ must have bit energy $X^2 \cdot E_b$, and corresponding SNR equal to $\frac{X^2 \cdot E_b}{N_0}$ (with N_0 the single sided noise power density)

The BER for DPSK modulation with non-coherent detection over an AWGN channel with SNR E_b/N_0 is

$$P_b(\gamma) = \frac{1}{2} e^{-\frac{E_b}{N_0}} = \frac{1}{2} e^{-\gamma}$$, where $\gamma = \frac{E_b}{N_0}$ is the SNR.

The BER over the fading channel is the expected value $P_{b,f} = E[P_b(X^2\gamma)]$. This is calculated as

$$P_{b,f} = E[P_b(X^2\gamma)] = \int_{-\infty}^{\infty} f_X(x) \cdot P_b(x^2\gamma)\, dx = \frac{1}{a}\int_0^a \frac{1}{2} e^{-x^2\gamma} dx =$$

$$/y=\sqrt{2\gamma}x/ \quad = \frac{1}{2a}\frac{1}{\sqrt{2\gamma}}\int_0^{\sqrt{2\gamma}\cdot a} e^{-\frac{y^2}{2}} dy = \frac{\sqrt{2\pi}}{2a\sqrt{2\gamma}} \cdot \frac{1}{\sqrt{2\pi}}\int_0^{a\sqrt{2\gamma}} e^{-\frac{y^2}{2}} dy =$$

$$= \frac{\sqrt{\pi}}{2a\sqrt{\gamma}} \cdot \left(Q(0) - Q(a\sqrt{2\gamma}) \right)$$

If we set $\sqrt{\gamma_0} = a\sqrt{2\gamma}$, i.e. $\gamma_0 = a^2 2\gamma = a^2 \frac{2E_b}{N_0}$, then we get the BER to be:

$$P_{b,f} = \frac{\sqrt{\pi}}{2a\frac{\sqrt{\gamma_0}}{\sqrt{2}a}} \left(\frac{1}{2} - Q(\sqrt{\gamma_0})\right) = \sqrt{\frac{\pi}{2\gamma_0}} \left(\frac{1}{2} - Q(\sqrt{\gamma_0})\right)$$, with

$$\gamma_0 = a^2 \frac{2E_b}{N_0}$$

4.9 Modulation scheme: binary FSK

Transmission rate: $r = 2{,}5$ Mbit/s $= 2{,}5 \cdot 10^6$ bit/s $\Rightarrow$

Symbol (bit) time duration: $T_b = \frac{1}{r} = \frac{1}{2{,}5 \cdot 10^6} = 4 \cdot 10^{-7}$ s

The modulation alphabet consists of the signals:

$$S_i(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t + \theta_i), \quad 0 \le t \le T_b, \quad i = 0, 1, \ldots, M-1$$

The modulated signal is

$$s(t) = \sum_{k=-\infty}^{\infty} S_{a_k}(t - kT_b)$$

, where $\{a_k\}$ is the information bit (symbol) sequence.

The amplitude of the signal $s(t)$ is obviously $\sqrt{\frac{2E_b}{T_b}} = 1\,\mu V \Rightarrow$

$\Rightarrow \sqrt{\frac{2E_b}{T_b}} = 10^{-6}$ V $\Rightarrow E_b = \frac{10^{-12} T_b}{2} = 2 \cdot 10^{-19}\ V^2 s$

The SNR is then $\frac{E_b}{N_0} = \frac{2 \cdot 10^{-19}\ V^2 s}{4 \cdot 10^{-20}\ V^2/Hz} = 5\ Hz \cdot s = 5$

The bit error probability rates (BER) are calculated according to the formulas for the different cases:

a) Coherently detected 2-FSK:

$$P_b = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q(\sqrt{5}) \approx 1{,}2674 \cdot 10^{-2}$$

b) Coherently detected MSK:

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q(\sqrt{10}) \approx 7{,}827 \cdot 10^{-4}$$

c) Non-coherently detected 2-FSK

$$P_e = \frac{1}{2} \exp\left(-\frac{E_b}{2N_0}\right) = \frac{1}{2} e^{-2{,}5} \approx 4{,}1042 \cdot 10^{-2}$$

4.10 Modulation scheme: binary FSK

$P_{e,\text{non-coherent}} = 20\% = 0{,}2$

The formula for the BER of non-coherent binary FSK over Rayleigh fading channel is

$$P_{e,\text{non-coherent}} = \frac{1}{2+\gamma_0} = 0{,}2 \implies \gamma_0 = \frac{0{,}6}{0{,}2} = 3$$

a) The BER for coherent detection of binary FSK over Rayleigh fading channel of mean SNR γ_0 is

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{2+\gamma_0}}\right) = \frac{1}{2}\left(1 - \sqrt{0{,}6}\right) \approx 1{,}127 \cdot 10^{-1}$$

b) The BER for binary DPSK is

$$P_e = \frac{1}{2(1+\gamma_0)} = \frac{1}{8} = 0{,}125$$

c) In this case we obtain the mean SNR γ_0 from

$$\frac{1}{2+\gamma_0} = 0{,}05 \implies \gamma_0 = 18$$

The BER for coherent BFSK is

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{2+\gamma_0}}\right) = \frac{1}{2}\left(1 - \sqrt{0{,}9}\right) \approx 2{,}5658 \cdot 10^{-2}$$

The BER for binary DPSK is

$$P_e = \frac{1}{2(1+\gamma_0)} = \frac{1}{38} \approx 2{,}6316 \cdot 10^{-2}$$

The difference in BER is approximately $6{,}58 \cdot 10^{-4}$, i.e. it is almost negligible.

4.11 a) Since DPSK is a form of PSK we have that its power density spectrum (PDS) is

$$\Phi_{\tilde{s}}(f) = 2E_b \operatorname{sinc}^2(fT_b)$$

The maximum of $\Phi_{\tilde{s}}(f)$ is obtained at $f=0$ and is $2E_b$. According to the condition we have

$$\left[\frac{2E_b \operatorname{sinc}^2(fT_b)}{2E_b}\right]_{dB} \le -40 \Rightarrow \operatorname{sinc}^2(fT_b) \le 10^{-4}$$

The $\operatorname{sinc}^2(t)$ function has it local maxima at points t_0 of the type $t_0 = k+\frac{1}{2}$, where k is integer. The value of the local maximum at $t_0 = k+\frac{1}{2}$ is $\operatorname{sinc}^2(t_0) = \left(\frac{1}{\pi(k+\frac{1}{2})}\right)^2$. Requiring this value to be at most 10^{-4} we get

$$\left(\frac{1}{\pi(k+\frac{1}{2})}\right)^2 \le 10^{-4} \Rightarrow \pi(k+\tfrac{1}{2}) \ge 10^2 \Rightarrow k \ge \frac{100}{\pi} - \frac{1}{2} \approx 31{,}3 \Rightarrow k \ge 32$$

This means that $\operatorname{sinc}^2(31{,}5) \ge 10^{-4} > \operatorname{sinc}^2(32{,}5)$. The actual value of t for which $\operatorname{sinc}^2(t) = 10^{-4}$ and $t \in [31{,}5, 32{,}5]$ is difficult to determine analytically. An upper bound on this value is $t=32$. Thus the requirement is satisfied if $f_{max} \cdot T_b \ge 32$. Here $f_{max} = \frac{B}{2} = \frac{25 \cdot 10^3}{2} = 1{,}225 \cdot 10^3$ and thus $T_b \ge \frac{32 \cdot 2}{25 \cdot 10^3} = 256 \cdot 10^{-5} \Rightarrow r = \frac{1}{T_b} \le \frac{10^5}{256} \approx 390{,}62$ bit/s.

b) In this case the PDS is $\Phi_{\tilde{s}}(f) = \frac{32E_b}{\pi}\left[\frac{\cos(2\pi f T_b)}{1-(4fT_b)^2}\right]^2$, which gives the condition $\left|\frac{\cos(2\pi f T_b)}{1-(4fT_b)^2}\right| \le 10^{-2}$. Numerical estimation gives $fT_b \gtrsim 2{,}5 \Rightarrow T_s \ge \frac{2{,}5}{f_{max}} = \frac{2 \cdot 2{,}5}{25 \cdot 10^3} = 2 \cdot 10^{-4}$. For the rate we have $r = \frac{1}{T_b} \le 5 \cdot 10^3$ bit/s $= 5$ kbit/s

c) From figure 4.28 (page 248) we can read out that $fT_b \gtrsim 0{,}75 \Rightarrow$
$\Rightarrow T_b \ge \frac{3}{4 \cdot f_{max}} = \frac{3 \cdot 2}{4 \cdot 25 \cdot 10^3} = 6 \cdot 10^{-5}$. The maximum rate is $r = \frac{1}{T_b} = \frac{10^5}{6} \approx 1{,}67 \cdot 10^4$ bit/s $= 16{,}7$ kbit/s

4.12 The mean-SNR γ_0 is computed from the corresponding formulas for the BER over slowly, Rayleigh fading channel.

a) Coherently detected PSK (binary)

$$10^{-5} \geq P_e = \frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{1+\gamma_0}}\right) \Rightarrow \sqrt{\frac{\gamma_0}{1+\gamma_0}} \geq 1-2\cdot 10^{-5} \Rightarrow$$

$$\Rightarrow \frac{\gamma_0}{1+\gamma_0} \gtrsim 0{,}99996 \Rightarrow \gamma_0 \gtrsim \frac{0{,}99996}{0{,}00004} = 24999$$

In logarithmic scale we have

$[\gamma_0]_{dB} \gtrsim 43{,}979$ dB

b) Coherently detected FSK (binary)

$$10^{-5} \geq P_e = \frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{2+\gamma_0}}\right) \Rightarrow \sqrt{\frac{\gamma_0}{2+\gamma_0}} \gtrsim 0{,}99996 \Rightarrow \gamma_0 \gtrsim 2\,\frac{0{,}99996}{0{,}00004} = 49998$$

In logarithmic units:

$[\gamma_0]_{dB} \gtrsim 46{,}99$ dB

c) Non-coherently detected FSK (binary)

$$10^{-5} \geq P_e = \frac{1}{2+\gamma_0} \Rightarrow \gamma_0 \geq 10^5 - 2 = 99998 \approx 10^5.$$

In logarithmic units:

$[\gamma_0]_{dB} \gtrsim 50$ dB