Digital Communication, Continuation Course

Power Spectral Density of Digital Modulation Schemes

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Power Spectral Density of Digital Modulation Schemes

We would like to determine the power spectral density (PSD) of a digitally modulated signal. We are interested both in a general expression of the PSD and in explicit expressions for the standard signal constellations that we have considered previously. To determine an expression of the PSD of digital modulation, we need to consider not just one signal interval as in the course TSKS01 Digital Communication. Instead we consider the input to be an infinite sequence $A[n]$ of symbols. For the explicit expressions for standard signal constellations, we will assume that those symbols are independent and identically distributed and that the basis functions are orthonormal.

A.1 Multi-Dimensional Pulse-Amplitude Modulation

Consider a wide-sense stationary process which is an infinite sequence $A[n]$ of symbols from the symbol alphabet used ($n = -\infty, \ldots, +\infty$). We refer to this process as the information process. This sequence is first mapped onto an infinite sequence $S[n]$ of $N$-dimensional signal vectors. Each dimension in this vector sequence is a real-valued time-discrete stochastic process $S_i[n]$ of its own, which we will refer to as a component process. Since the information process is wide-sense stationary, then the component processes are jointly stationary in the wide sense. These vectors are then modulated using the $N$ basis functions. For the $n$-th vector in that sequence, we use the time-shifted (real-valued) basis functions $\{\phi_i(t - nT - \Psi)\}_{i=1}^N$. Here $\Psi$ is a random delay, independent of $A[n]$ for all $n$, and uniformly distributed over the signalling interval $[0, T)$. This random delay is included to maintain the wide-sense stationarity, despite the periodic structure that modulated signal

actually has. The resulting signal is then

\[ S(t) = \sum_n \sum_{i=1}^N S_i[n] \phi_i(t - nT - \Psi), \]  

(A.1)

that is, the component processes \( S_i[n] \) are pulse-amplitude modulated using their basis functions as pulse shapes and then added to form \( S(t) \). Note that we are using the same \( \Psi \) in all dimensions to maintain orthogonality among the basis functions.

### A.1.1 General expressions

We will first compute the autocorrelation function (ACF) of (A.1) and then use it to compute the PSD. We can write the ACF of the resulting signal \( S(t) \) as

\[ r_S(\tau) = E\left\{ S(t + \tau) S^*(t) \right\} \]

\[ = E \left\{ \sum_n \sum_{i=1}^N S_i[n] \phi_i(t + \tau - nT - \Psi) \sum_m \sum_{l=1}^N S_l^*[m] \phi_l^*(t - mT - \Psi) \right\} \]

\[ = E \left\{ \sum_n \sum_m \sum_i \sum_{l=1}^N S_i[n] S_l^*[m] \phi_i(t + \tau - nT - \Psi) \phi_l^*(t - mT - \Psi) \right\}. \]

We use the linearity of the expectation to rewrite that as

\[ r_S(\tau) = \sum_n \sum_m \sum_i \sum_{l=1}^N E \left\{ S_i[n] S_l^*[m] \phi_i(t + \tau - nT - \Psi) \phi_l^*(t - mT - \Psi) \right\}. \]

(A.2)

From the assumption that the random delay \( \Psi \) and the information process \( A[n] \) are independent of each other we find that also the signals \( S_i[n] \) are independent of the delay. Then we get

\[ r_S(\tau) = \sum_n \sum_m \sum_{i=1}^N \sum_{l=1}^N E \left\{ S_i[n] S_l^*[m] \right\} E \left\{ \phi_i(t + \tau - nT - \Psi) \phi_l^*(t - mT - \Psi) \right\}. \]

We identify the first expectation in (A.2) as the cross correlation of two component processes. Recall that the component processes are jointly stationary in the wide sense. Thus, this cross correlation is only a function of \( n - m \). We denote this as \( E \left\{ S_i[n] S_l^*[m] \right\} = r_{S_i,S_l}[n - m] \). The second expectation (A.2) is the expectation of a function of \( \Psi \). Since \( \Psi \) is uniformly distributed over \([0, T]\), its probability density is \( 1/T \) in that interval, and zero elsewhere. These observations give us

\[ r_S(\tau) = \sum_n \sum_m \sum_{i=1}^N \sum_{l=1}^N r_{S_i,S_l}[n - m] \int_0^T \phi_i(t + \tau - nT - \psi) \phi_l^*(t - mT - \psi) \frac{1}{T} \, d\psi. \]
A.1. Multi-Dimensional Pulse-Amplitude Modulation.

We introduce the new variables $k = n - m$ and $u = t + \tau - nT - \psi$. Then we get

$$r_S(\tau) = \frac{1}{T} \sum_{k} \sum_{i=1}^{N} \sum_{l=1}^{N} r_{S_i,S_l}[k] \sum_{n} \int_{t-nT+\tau}^{t-nT+\tau} \phi_i(u)\phi_i^*(u - \tau + kT) \, du$$

$$= \frac{1}{T} \sum_{k} \sum_{i=1}^{N} \sum_{l=1}^{N} r_{S_i,S_l}[k] \int_{-\infty}^{\infty} \phi_i(u)\phi_i^*(u - \tau + kT) \, du. \quad (A.3)$$

This is as far as we can get in computing the ACF, without making further assumptions around the correlation between the symbols in the information sequence and around the basis functions.

Using the general ACF expression in (A.3), the PSD of $S(t)$ is the Fourier transform of $r_S(\tau)$. We get

$$R_S(f) = \int_{-\infty}^{\infty} \frac{1}{T} \sum_{k} \sum_{i=1}^{N} \sum_{l=1}^{N} r_{S_i,S_l}[k] \int_{-\infty}^{\infty} \phi_i(u)\phi_i^*(u - \tau + kT) \, du \, e^{-j2\pi f \tau} \, d\tau.$$ 

In the inner integral, we let $v = u - \tau + kT$ and then we have

$$R_S(f) = \frac{1}{T} \int_{-\infty}^{\infty} \sum_{k} \sum_{i=1}^{N} \sum_{l=1}^{N} r_{S_i,S_l}[k] \int_{-\infty}^{\infty} \phi_i(u)\phi_i^*(v) e^{-j2\pi(kT+u-v)f} \, du \, dv$$

$$= \frac{1}{T} \sum_{i=1}^{N} \sum_{l=1}^{N} \sum_{k} r_{S_i,S_l}[k] e^{-j2\pi kTf} \int_{-\infty}^{\infty} \phi_i(u) e^{-j2\pi u f} \, du \int_{-\infty}^{\infty} \phi_i^*(v) e^{j2\pi v f} \, dv. \quad (A.4)$$

We identify the two integrals at the second line of (A.4) as the continuous-time Fourier transform of $\phi_i(t)$ and $\phi_i^*(-t)$, respectively, and the innermost sum as the discrete-time Fourier transform of $r_{S_i,S_l}[k]$. Hence, we have

$$R_S(f) = \frac{1}{T} \sum_{i=1}^{N} \sum_{l=1}^{N} R_{S_i,S_l}[fT] \Phi_i(f) \Phi_i^*(f). \quad (A.5)$$

Notice that we can also express this formula as the quadratic form

$$R_S(f) = \frac{1}{T} \begin{pmatrix} \Phi_1(f), \ldots, \Phi_N(f) \end{pmatrix} \begin{pmatrix} R_{S_1,S_1}[fT] & \cdots & R_{S_1,S_N}[fT] \\ \vdots & \ddots & \vdots \\ R_{S_N,S_1}[fT] & \cdots & R_{S_N,S_N}[fT] \end{pmatrix} \begin{pmatrix} \Phi_1^*(f) \\ \vdots \\ \Phi_N^*(f) \end{pmatrix}. \quad (A.6)$$

This PSD expression does not hold only for digital modulation. It is a general expression for simultaneous pulse-amplitude modulation of several stochastic processes using possibly different pulse shapes for each process. We stress that the basis function does not need to be orthonormal in order for (A.5) and (A.6) to hold.
Real-valued signals

In most cases we are only interested in real-valued signals. Assuming that all involved processes are real-valued, then their PSDs have to be real as well. Nevertheless, many of the terms in (A.5) has non-zero imaginary parts, which means that all the imaginary parts of the terms \( R_{S_i,S_l}[fT] \Phi_i(f) \Phi_l^*(f) \) have to cancel out in the summation. Actually, they cancel out in pairs, which can be realized by observing that the Fourier transforms of the basis functions satisfy \( \Phi_i(f) \Phi_l^*(f) = (\Phi_l(f) \Phi_i^*(f))^* \) and that the cross-correlations of the symbols satisfy \( R_{S_i,S_l}[\theta] = R_{S_l,S_i}[\theta] \). Therefore, the imaginary part of the mixed term \( R_{S_i,S_l}[fT] \Phi_i(f) \Phi_l^*(f) \) cancels out the imaginary part of \( R_{S_l,S_i}[fT] \Phi_l(f) \Phi_i^*(f) \). The terms on the main diagonal of the matrix in (A.6) are all real and positive. This observation will have implications further on.

A.1.2 Important Special Cases

Independent symbols

As stated in the introduction, we would like to express the PSDs of standard choices of signal constellations and basis functions, for the simple case where the information process \( A[n] \) consists of independent and identically distributed variables. This is a realistic assumption when the information process describes data that has been source coded to remove the correlation that might have existed in the original source. Next, we would like to determine all involved cross spectra \( R_{S_i,S_l}[fT] \) for this assumed situation. We should notice that the assumption implies that the information process is stationary in the strict sense. The mapping to the vector sequence \( \tilde{S}[n] \) then results in component processes \( S_i[n] \), \( i \in \{1, 2, \ldots, N\} \), that are jointly stationary in the strict sense. Moreover, the independence of the symbols in the information process results in that \( S_i[n_1] \) and \( S_l[n_2] \) are independent if \( n_1 \neq n_2 \) holds regardless of \( i \) and \( l \). This means that the power spectral densities and cross spectra of the component processes become particularly simple.

Let \( \lambda_{S_i,S_l}[k] = E\{(S_i[n+k] - m_{S_i})(S_l[n] - m_{S_l})^*\} \) denote the cross covariance of the two component processes and let \( m_{S_i} = E\{S_i[n]\} \) denote the mean value. Then the cross correlation is

\[
r_{S_i,S_l}[k] = E\{S_i[n+k]S_l[n]^*\} = \lambda_{S_i,S_l}[k] + m_{S_i}m_{S_l} = \lambda_{S_i}[n],S_l[n]\delta[k] + m_{S_i}m_{S_l}, \tag{A.7}
\]

where \( \lambda_{S_i}[n],S_l[n] = \lambda_{S_i,S_l}[0] \) is the covariance of the stochastic variables \( S_i[n] \) and \( S_l[n] \). By taking the Fourier transform of (A.7) we have the cross spectrum

\[
R_{S_i,S_l}[\theta] = \lambda_{S_i}[n],S_l[n] + m_{S_i}m_{S_l} \sum_m \delta(\theta - m). \tag{A.8}
\]
A.1. Multi-Dimensional Pulse-Amplitude Modulation.

Thus, all we need to do to determine those cross spectra is to analyze the cases \( k = 0 \) and \( k \neq 0 \) separately. An important observation that we can make is that these cross spectra are all real-valued. We have observed the general relation

\[
 r_{S_l, S_i}[k] = r_{S_i, S_l}[-k]
\]

which gives us

\[
 R_{S_l, S_i}[\theta] = R_{S_i, S_l}[\theta].
\]

But as noted, the imaginary parts are zero, so we have

\[
 R_{S_l, S_i}[\theta] = R_{S_i, S_l}[\theta].
\]

### Uncorrelated dimensions

Next, we make another set of assumption namely that the components \( S_l[n + k] \) and \( S_i[n] \) are uncorrelated for \( i \neq l \) for all \( k \) and that they have zero mean \( \mathbb{E}\{S_i\} = 0 \) for all \( i \). In this special case, the cross correlation is

\[
 r_{S_l, S_i}[k] = \mathbb{E}\{S_l[n + k]S_i[n]\} = \begin{cases} r_{S_l}[k] & \text{if } i = l, \\ \mathbb{E}\{S_l[n + k]\}\mathbb{E}\{S_i[n]\} & \text{if } i \neq l. \end{cases} \tag{A.9}
\]

If we apply this property to (A.6), then the general expression of the PSD collapses to

\[
 R_S(f) = \frac{1}{T} \sum_{i=1}^{N} R_{S_i}[fT] |\Phi_i(f)|^2
\]

since all terms in the matrix then are zero except on the main diagonal, where we actually have the power spectral densities \( r_{S_i}[k] \) of the component processes.

### Special basis functions

Again we assume that consecutive symbols are independent. Our observations that the imaginary parts of the mixed term \( R_{S_l, S_i}[fT]\Phi_i(f)\Phi_i^*(f) \) cancels out the imaginary part of \( R_{S_i, S_l}[fT]\Phi_i(f)\Phi_i^*(f) \) and that the cross spectra \( R_{S_l, S_i}[\theta] \) are real-valued have important implications that help us identify another interesting special case.

If \( \Phi_i(f)\Phi_i^*(f) \) is purely imaginary for all \( i \neq l \) then we have

\[
 \Phi_i(f)\Phi_i^*(f) = -\Phi_i(f)\Phi_i^*(f), \tag{A.10}
\]

and all mixed terms in (A.5) cancel out. Once again, the general PSD expression collapses to

\[
 R_S(f) = \frac{1}{T} \sum_{i=1}^{N} R_{S_i}[fT] |\Phi_i(f)|^2. \tag{A.11}
\]
A.1.3 Sine-Shaped Basis Functions

Our final goal is to derive explicit expressions for the PSDs of standard signal constellations. Since the general PSD expression in (A.5) contains the Fourier transforms of the basis functions, we need to compute them for the standard choices of basis functions; namely sine-shaped basis functions with $N = 2$.

**Same frequency**

Consider the two basis functions

\[ \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_1 t), \quad \text{for} \ 0 \leq t < T, \]

\[ \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_1 t), \quad \text{for} \ 0 \leq t < T, \]

and assume, as usual, that the $2f_1T$ is a positive integer since this guarantees that the two basis functions are orthonormal. By noting that $\phi_1(t), \phi_2(t)$ are the product of a sinusoidal signal and a rectangular box (with support between 0 and $T$, we can use standard relations for the Fourier transform and the basis functions have the Fourier transforms

\[ \Phi_1(f) = \sqrt{\frac{T}{2}} e^{-j\pi f T} \left( j^{-2f_1T} \text{sinc}((f + f_1)T) + j^{2f_1T} \text{sinc}((f - f_1)T) \right), \]

\[ \Phi_2(f) = j \sqrt{\frac{T}{2}} e^{-j\pi f T} \left( j^{-2f_1T} \text{sinc}((f + f_1)T) - j^{2f_1T} \text{sinc}((f - f_1)T) \right). \]

Note that we have utilized that $e^{j\pi f_1T} = (e^{j\pi/2})^{2f_1T} = j^{2f_1T}$ since $2f_1T$ is an integer.

We can see that we have

\[ \Phi_1(f)\Phi_2^*(f) = -j \frac{T}{2} \left( \text{sinc}^2((f + f_1)T) - \text{sinc}^2((f - f_1)T) \right), \]

which obviously is purely imaginary. Thus, the condition in (A.10) is satisfied and in this two-dimensional situation the simplified PSD expression in (A.11) becomes

\[ R_S(f) = \frac{1}{T} \left( R_{S_1}[fT]\Phi_1(f)\right|_{fT}^2 + R_{S_2}[fT]\Phi_2(f)\right|_{fT}^2 \]

for the case of independent symbols. Moreover, we have

\[ |\Phi_1(f)|^2 = \frac{T}{2} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T} \text{sinc}((f - f_1)T) \right)^2, \]

\[ |\Phi_2(f)|^2 = \frac{T}{2} \left( \text{sinc}((f + f_1)T) - (-1)^{2f_1T} \text{sinc}((f - f_1)T) \right)^2. \]
A.1. Multi-Dimensional Pulse-Amplitude Modulation.

Figure A.1: Illustration of the energy spectra $|\Phi_1(f)|^2$ and $|\Phi_2(f)|^2$ in (A.14) and (A.15), which are sine-based basis functions with same frequency.

Most often the carrier frequency $f_1$ is much larger than $1/T$ (which is proportional to the signal bandwidth). Consequently, $2f_1 T$ is a large integer and the mixed terms in (A.14) and (A.15) will be small compared to the squared terms, and we can use the approximation

$$|\Phi_1(f)|^2 \approx \frac{T}{2} \left( \text{sinc}^2 \left( (f + f_1) T \right) + \text{sinc}^2 \left( (f - f_1) T \right) \right) \approx |\Phi_2(f)|^2.$$

Based on this observation, we deduce that the bandwidth of a two-dimensional signal constellation using these basis functions is roughly the same as the bandwidth of a one-dimensional signal constellation using one of those basis functions. This is illustrated in Figure A.1, which shows the spectra $|\Phi_1(f)|^2$ and $|\Phi_2(f)|^2$ around the positive carrier frequency $f_1$. Since these spectra are overlapping, it is very common to count pairs of dimensions instead of just dimensions when counting the bandwidth content. Note that each energy spectrum takes the form of a squared sinc-function, which theoretically has infinite bandwidth. However, 90% of the energy lies within the main lobe, which has a width of $2/T$, and this is a common approximation of the bandwidth.
Different frequencies

For basis functions using different frequencies, the situation is a bit more complicated than the above, as we will see next. Let $2f_1T$ and $2f_2T$ be different positive integers, which again guarantees that the two basis functions

$$
\phi_1(t) = \sqrt{\frac{T}{2}} \cos(2\pi f_1 t), \quad \text{for } 0 \leq t < T,
$$

$$
\phi_2(t) = \sqrt{\frac{T}{2}} \cos(2\pi f_2 t), \quad \text{for } 0 \leq t < T,
$$

are orthonormal. Similar to (A.12) (using standard relations for the Fourier transform), we find that these signals have the Fourier transforms

$$
\Phi_1(f) = \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( j^{-2f_1T} \text{sinc}((f + f_1)T) + j^{2f_1T} \text{sinc}((f - f_1)T) \right),
$$

$$
\Phi_2(f) = \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( j^{-2f_2T} \text{sinc}((f + f_2)T) + j^{2f_2T} \text{sinc}((f - f_2)T) \right).
$$

We obviously have

$$
\Phi_1(f) \Phi_2^*(f) = \frac{T}{2} \left( j^{-2f_1T+2f_2T} \text{sinc}((f + f_1)T) \text{sinc}((f + f_2)T) \right.
\left. + j^{-2f_1T-2f_2T} \text{sinc}((f + f_1)T) \text{sinc}((f - f_2)T) \right.
\left. + j^{2f_1T+2f_2T} \text{sinc}((f - f_1)T) \text{sinc}((f + f_2)T) \right.
\left. + j^{2f_1T-2f_2T} \text{sinc}((f - f_1)T) \text{sinc}((f - f_2)T) \right).
$$

(A.16)

If both $2f_1T$ and $2f_2T$ are even or both are odd, then the exponents $-2f_1T + 2f_2T$, $-2f_1T - 2f_2T$, $2f_1T + 2f_2T$ and $2f_1T - 2f_2T$ are all even. Consequently, all those powers of $j$ are real, and the complete expression of $\Phi_1(f) \Phi_2^*(f)$ is real. If instead one of $2f_1T$ and $2f_2T$ is odd and the other even, then all those exponents are odd, and all those powers of $j$ are imaginary and the whole expression of $\Phi_1(f) \Phi_2^*(f)$ is purely imaginary. Thus, still assuming independent symbols, in the first case (both even or both odd) we have the general PSD expression

$$
R_S(f) = \frac{1}{T} \left( R_{S_1}[fT] |\Phi_1(f)|^2 + 2R_{S_1,S_2}[fT] \Phi_1(f) \Phi_2^*(f) + R_{S_2}[fT] |\Phi_2(f)|^2 \right) \tag{A.17}
$$

for this two-dimensional situation. In the second case, though, we have the PSD

$$
R_S(f) = \frac{1}{T} \left( R_{S_1}[fT] |\Phi_1(f)|^2 + R_{S_2}[fT] |\Phi_2(f)|^2 \right) \tag{A.18}
$$

for this two-dimensional situation according to our observations in Section A.1.2. Note that (A.18) can also be computed directly from (A.17), by observing that $\text{Re} \{\Phi_1(f) \Phi_2^*(f)\}$ is
zero in the latter case. We can also observe that if \(2f_1T\) and \(2f_2T\) are fairly large integers, then the second and third term in (A.16) containing
\[
\sin((f + f_1)T)\sin((f - f_2)T) \quad \text{and} \quad \sin((f - f_1)T)\sin((f + f_2)T)
\]
are small compared to the two other terms containing
\[
\sin((f + f_1)T)\sin((f + f_2)T) \quad \text{and} \quad \sin((f - f_1)T)\sin((f - f_2)T)
\]
in the expression of \(\Phi_1(f)\Phi_2^*(f)\) above, since the offsets between the \(\sin\) functions are smaller. Therefore, we can approximate (A.16) as
\[
\Phi_1(f)\Phi_2^*(f) \approx \frac{T}{2} \left( j^{-2f_1T-2f_2T}\sin((f + f_1)T)\sin((f + f_2)T) \\
+ j^{2f_1T+2f_2T}\sin((f - f_1)T)\sin((f - f_2)T) \right).
\]

Furthermore, if the integers \(2f_1T\) and \(2f_2T\) are fairly far apart, then we can also claim that those two remaining parts are small and we can even disregard \(\Phi_1(f)\Phi_2^*(f)\) altogether, regardless of which of the two observed cases we have.

Similar to the case with same frequency, we have the energy spectra
\[
|\Phi_1(f)|^2 = \frac{T}{2} \left( \sin((f + f_1)T) + (-1)^{2f_1T}\sin((f - f_1)T) \right)^2, \quad (A.19)
\]
\[
|\Phi_2(f)|^2 = \frac{T}{2} \left( \sin((f + f_2)T) + (-1)^{2f_2T}\sin((f - f_2)T) \right)^2, \quad (A.20)
\]
of the basis functions, and again we can use the approximations
\[
|\Phi_1(f)|^2 \approx \frac{T}{4} \left( \sin^2((f + f_1)T) + \sin^2((f - f_1)T) \right)
\]
\[
|\Phi_2(f)|^2 \approx \frac{T}{4} \left( \sin^2((f + f_2)T) + \sin^2((f - f_2)T) \right)
\]
if \(2f_1T\) and \(2f_2T\) are fairly large integers. To keep the total bandwidth of the signal low, we want the two carrier frequencies to be as close to each other as possible, but still chosen such that \(2f_1T\) and \(2f_2T\) are integers to guarantee orthogonality. Thus, we could choose them to fulfill \(2f_1T = 2f_2T \pm 1\), or equally that \(f_1 = f_2 \pm 1/(2T)\).

**Comparison of Energy Spectra**

Next, we illustrate and compare the energy spectra \(|\Phi_1(f)|^2\) and \(|\Phi_2(f)|^2\) in (A.19) and (A.20), respectively. Recall that these represented two cosine basis functions with different frequencies: \(f_1\) and \(f_2\). Figure A.2 shows these spectra at positive carrier frequencies, assuming a difference of \(1/T\) between \(f_1\) and \(f_2\); that is: \(f_2 = f_1 + 1/T\). Note that this
carrier separation is sufficient for having one cosine and one sine basis function at each of the frequencies and still guarantee orthogonality between all four of them.

Each energy spectra in Figure A.2 has the form of a squared sinc-function. Around 90% of the energy lies within the main lobe, which has a width of $2/T$ around the carrier frequency. Hence, the approximate bandwidth of the linearly modulated signal sent over one of the basis function is $2/T$. Recall that the Nyquist criterion prescribes that a bandwidth of $1/T$ is sufficient for a pulse-amplitude modulated (PAM) passband signal with symbol duration $T$, which is substantially smaller bandwidth than what we obtained here. This is because we use a box-shaped pulse (multiplied with a cosine) instead of a sinc pulse.

Interestingly, the energy spectra of the two basis functions overlap substantially in Figure A.2, although the basis functions are orthogonal. In fact, even the main lobes overlap by 50%. This has the important consequence that if we utilize $K$ cosine basis functions with
frequencies \( f_k = f_1 + (k-1)/T \), then the total bandwidth of the signal will not be \( K \cdot (2/T) \) (i.e., adding each separate bandwidth) but rather \( (K+1)/T \). This is approximately equal to \( K/T \) if \( K \) is large and thus essentially the same as achieved by \( K \) ideal PAMs with sinc pulses. This principle is used in orthogonal frequency division multiplexing (OFDM) to obtain orthogonal basis functions and to have relatively good bandwidth utilization. At the order of \( K = 1000 \) carrier frequencies are common in such systems.

### A.2 One-Dimensional Constellations

For the one-dimensional case, \( N = 1 \), the general PSD expression in (A.5) simplifies to

\[
R_S(f) = \frac{1}{T} R_{S_1}[fT] |\Phi_1(f)|^2,
\]

which we recognize as the expression that holds for pulse-amplitude modulation of a time-discrete stochastic process \( S_1[n] \). This should not be a surprise, since we are using pulse-amplitude modulation in each dimension.

We assume that \( 2f_1T \) is a positive integer, as before. Here it is not needed to guarantee any orthogonality, since we only have one dimension. Instead we simply demand that, in order to make the calculation of the signal energy simple. We have seen previously that the standard choice of basis function

\[
\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_1 t), \quad \text{for } 0 \leq t < T
\]

has Fourier transform

\[
\Phi_1(f) = \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( -2f_1 T \text{sinc}((f + f_1)T) + j2f_1 T \text{sinc}((f - f_1)T) \right),
\]

and energy spectrum

\[
|\Phi_1(f)|^2 = \frac{T}{2} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1 T} \text{sinc}((f - f_1)T) \right)^2.
\]

For an information process that consists of independent and identically distributed variables, we have seen in (A.8) that the PSD of a component process is

\[
R_{S_1}[\theta] = \sigma_{S_1}^2 + m_{S_1}^2 \sum_m \delta(\theta - m).
\]  

(A.21)

If \( 2f_1T \) is even, such that \( f_1T \) is also an integer, then \( |\Phi_1(f)|^2 = 0 \) for all integer values of \( fT \), except for \( f = \pm f_1 \) which are the centers of the sinc-functions. Hence, only two
of the impulses in $R_{S_1}[fT]$ survive the multiplication with $|\Phi_1(f)|^2$, namely those at $\pm f_1$. All other impulses in $R_{S_1}[fT]$ coincide with zeros in $|\Phi_1(f)|^2$. The resulting PSD is

$$R_S(f) = \frac{\sigma^2_{S_1}}{2} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T}\text{sinc}((f - f_1)T) \right)^2 + \frac{m^2_{S_1}}{2} \left( \delta((f + f_1)T) + \delta((f - f_1)T) \right) \quad (A.22)$$

Notice that $(-1)^{2f_1T} = 1$ in this particular case, thus (A.22) can be further simplified.

If on the other hand $2f_1T$ is odd, then no impulses in $R_{S_1}[fT]$ coincide with zeros in $|\Phi_1(f)|^2$. All impulses in $R_{S_1}[fT]$ survive, but will be differently scaled. The dominating impulses are those near $\pm f_1$, just as in the even case.

We note that if $m_{S_1} = 0$, then the impulses in (A.21) vanish and we obtain

$$R_S(f) = \frac{\sigma^2_{S_1}}{2} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T}\text{sinc}((f - f_1)T) \right)^2 + \frac{E_{\text{max}}}{4} \left( \delta((f + f_1)T) + \delta((f - f_1)T) \right) \quad (A.23)$$

for both even and odd values of $2f_1T$.

In the following, let $E_{\text{max}}$ and $E_{\text{avg}}$ denote the largest signal energy and the average signal energy, respectively, in the considered signal constellation. In the average case, we assume that all signals are equally probable.

### A.2.1 On-Off Keying (OOK)

Recall that the vectors used in on-off keying are $(0)$ and $(\sqrt{E_{\text{max}}})$. Assuming independent and equally probable symbols, we get the mean $m_{S_1} = \sqrt{E_{\text{max}}}/2$ and the variance $\sigma^2_{S_1} = E_{\text{max}}/4$. Here we have $E_{\text{avg}} = E_{\text{max}}/2$. Since the mean is non-zero, we only want to study the case where $2f_1T$ is even, to avoid the problems that we observed above for odd $2f_1T$.

In this case, we can use (A.22) and immediately obtain the PSD

$$R_S(f) = \frac{E_{\text{max}}}{8} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T}\text{sinc}((f - f_1)T) \right)^2 + \frac{E_{\text{max}}}{8} \left( \delta((f + f_1)T) + \delta((f - f_1)T) \right).$$

Expressed in the average signal energy, we have

$$R_S(f) = \frac{E_{\text{avg}}}{4} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T}\text{sinc}((f - f_1)T) \right)^2 + \frac{E_{\text{avg}}}{4} \left( \delta((f + f_1)T) + \delta((f - f_1)T) \right).$$
A.2.2 Binary Phase-Shift Keying (BPSK)

In BPSK we have the vectors $(±\sqrt{E_{\text{max}}})$ as constellation points. Assuming independent and equally probable symbols, we get the mean $m_{S_1} = 0$ and the variance $E_{\text{max}}$. Here we have $E_{\text{avg}} = E_{\text{max}}$. Since the mean is zero, we can allow $2f_1T$ to be any positive integer. Plugging the above into the one-dimensional PSD expression in (A.23), we get

$$R_S(f) = \frac{E_{\text{max}}}{2} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T}\text{sinc}((f - f_1)T) \right)^2.$$

Expressed in the average signal energy, we have

$$R_S(f) = \frac{E_{\text{avg}}}{2} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T}\text{sinc}((f - f_1)T) \right)^2.$$

A.2.3 Amplitude-shift keying (ASK)

Many different ASK constellations can be considered. Here we focus at 4-ASK, in which we have the vectors $(±\sqrt{E_{\text{max}}})$ and $(±\sqrt{E_{\text{max}}}/3)$ as constellation points. Assuming independent and equally probable symbols, we get the mean $m_{S_1} = 0$ and the variance $\frac{5}{9}E_{\text{max}}$. Here we have $E_{\text{avg}} = \frac{5}{9}E_{\text{max}}$. Since the mean is zero, we can allow $2f_1T$ to be any positive integer. Plugging the above into the one-dimensional PSD expression (A.23), we get

$$R_S(f) = \frac{5E_{\text{max}}}{18} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T}\text{sinc}((f - f_1)T) \right)^2.$$

Expressed in the average signal energy, we have

$$R_S(f) = \frac{E_{\text{avg}}}{2} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T}\text{sinc}((f - f_1)T) \right)^2.$$

A.3 Two-Dimensional Constellations using One Carrier Frequency

Next, we consider the two-dimensional case with the two same-frequency basis functions

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_1 t), \quad \text{for } 0 \leq t < T,$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_1 t), \quad \text{for } 0 \leq t < T,$$
where $2f_1T$ is a positive integer. We have seen that the general PSD expression in A.6 simplifies into

$$R_s(f) = \frac{1}{T} \left( R_{S_1}[fT] |\Phi_1(f)|^2 + R_{S_2}[fT] |\Phi_2(f)|^2 \right)$$

for these basis functions. We have also seen in Section A.1.3 that these basis functions have Fourier transforms

$$\Phi_1(f) = \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( j^{-2f_1T} \text{sinc}\left((f + f_1)T\right) + j^{2f_1T} \text{sinc}\left((f - f_1)T\right) \right),$$

$$\Phi_2(f) = j \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( j^{-2f_1T} \text{sinc}\left((f + f_1)T\right) - j^{2f_1T} \text{sinc}\left((f - f_1)T\right) \right),$$

and energy spectra

$$|\Phi_1(f)|^2 = \frac{T}{2} \left( \text{sinc}\left((f + f_1)T\right) + (-1)^{2f_1T} \text{sinc}\left((f - f_1)T\right) \right)^2,$$

$$|\Phi_2(f)|^2 = \frac{T}{2} \left( \text{sinc}\left((f + f_1)T\right) - (-1)^{2f_1T} \text{sinc}\left((f - f_1)T\right) \right)^2.$$ 

For an information process that consists of independent and identically distributed variables, we have seen in (A.21) that the PSD of the component processes are

$$R_{S_1}[\theta] = \sigma_{S_1}^2 + m_{S_1}^2 \sum_m \delta(\theta - m), \quad (A.24)$$

$$R_{S_2}[\theta] = \sigma_{S_2}^2 + m_{S_2}^2 \sum_m \delta(\theta - m). \quad (A.25)$$

When multiplying these expressions with the Fourier transforms of the basis functions, the situation is the same as in the one-dimensional case when it comes to even and odd $2f_1T$. In particular, for even values of $2f_1T$ the resulting PSD is

$$R_s(f) = \frac{\sigma_{S_1}^2}{2} \left( \text{sinc}\left((f + f_1)T\right) + (-1)^{2f_1T} \text{sinc}\left((f - f_1)T\right) \right)^2$$

$$+ \frac{m_{S_1}^2}{2} \left( \delta((f + f_1)T) + \delta((f - f_1)T) \right)$$

$$+ \frac{\sigma_{S_2}^2}{2} \left( \text{sinc}\left((f + f_1)T\right) - (-1)^{2f_1T} \text{sinc}\left((f - f_1)T\right) \right)^2$$

$$+ \frac{m_{S_2}^2}{2} \left( \delta((f + f_1)T) + \delta((f - f_1)T) \right).$$

Again, if $2f_1T$ is odd, then no impulses in $R_{S_1}[fT]$ or $R_{S_2}[fT]$ coincide with zeros in $|\Phi_1(f)|^2$. All impulses in $R_s[fT]$ survive, but are differently scaled depending on how far from the center of sinc-function it is. The dominating impulses are those near $\pm f_1$.

We note that if $m_{S_1}$ and $m_{S_2}$ are zero, then we do not have any of those problems.
A.3.1 Non-Binary PSK

For an $M$-ary PSK, we have $M$ signal constellation points evenly distributed over a circle of radius $\sqrt{E_{\text{avg}}}$ with its center in the origin. Note that $E_{\text{avg}}$ is both the maximum and average energy. Both component processes have zero mean, and variance $E/2$. The resulting PSD is

$$R_S(f) = \frac{E_{\text{avg}}}{4} \left( \text{sinc}\left((f + f_1)T\right) + (-1)^2 f_1 T \text{sinc}\left((f - f_1)T\right) \right)^2$$

$$+ \frac{E_{\text{avg}}}{4} \left( \text{sinc}\left((f + f_1)T\right) - (-1)^2 f_1 T \text{sinc}\left((f - f_1)T\right) \right)^2$$

$$= \frac{E_{\text{avg}}}{2} \left( \text{sinc}^2\left((f + f_1)T\right) + \text{sinc}^2\left((f - f_1)T\right) \right).$$

A.3.2 Quadrature Amplitude Modulation (QAM)

There are QAM constellations of many different sizes. 4-QAM is identical to 4-PSK, thus we will instead consider 16-QAM in detail. In 16-QAM we have 16 points symmetrically placed on a rectangular grid. In each dimension, we have a 4-ASK modulation. Both component processes have zero mean, and variance $5E_{\text{max}}/18$. The resulting PSD is

$$R_S(f) = \frac{5E_{\text{max}}}{36} \left( \text{sinc}\left((f + f_1)T\right) + (-1)^2 f_1 T \text{sinc}\left((f - f_1)T\right) \right)^2$$

$$+ \frac{5E_{\text{max}}}{36} \left( \text{sinc}\left((f + f_1)T\right) - (-1)^2 f_1 T \text{sinc}\left((f - f_1)T\right) \right)^2$$

$$= \frac{5E_{\text{max}}}{18} \left( \text{sinc}^2\left((f + f_1)T\right) + \text{sinc}^2\left((f - f_1)T\right) \right).$$

Notice that we have $E_{\text{avg}} = \frac{5}{9} \cdot E_{\text{max}}$, just as in the case of 4-ASK. The PSD can also be expressed in terms of the average signal energy:

$$R_S(f) = \frac{E_{\text{avg}}}{2} \left( \text{sinc}^2\left((f + f_1)T\right) + \text{sinc}^2\left((f - f_1)T\right) \right).$$