Equations for the Raised Cosine and Square-Root Raised Cosine Shapes

1 Raised Cosine Spectrum

A family of spectra that satisfy the Nyquist Theorem is the raised cosine family whose spectra are

$$Z(f) = \begin{cases} T_s & 0 \le |f| \le \dfrac{1-\beta}{2T_s} \\ \dfrac{T_s}{2}\left\{1+\cos\left[\dfrac{\pi T_s}{\beta}\left(|f| - \dfrac{1-\beta}{2T_s}\right)\right]\right\} & \dfrac{1-\beta}{2T_s} \le |f| \le \dfrac{1+\beta}{2T_s} \\ 0 & |f| > \dfrac{1+\beta}{2T_s} \end{cases} \tag{1}$$

where the parameter roll-off factor β is a real number in the interval $0 \le \beta \le 1$ that determines the bandwidth of the the spectrum. Since the spectrum is zero for $|f| > \frac{1+\beta}{2T_s}$, the bandwidth of the baseband pulse is $\frac{1+\beta}{2T_s}$. For bandpass QAM modulation, the bandwidth is twice that:

$$\mathrm{BW} = \frac{1+\beta}{T_s} = (1+\beta)R_s \tag{2}$$

where R_s is the transmitted symbol rate. The ideal low-pass rectangular spectrum is the special case where $\beta = 0$ which has a passband bandwidth equal to the symbol rate.

The corresponding time domain signal is

$$z(t) = \frac{\cos\left(\pi\beta\dfrac{t}{T_s}\right)}{1-\left(2\beta\dfrac{t}{T_s}\right)^2} \times \frac{\sin \pi\dfrac{t}{T_s}}{\pi\dfrac{t}{T_s}} \tag{3}$$

Observe that $z(t)$ has zero-crossings at $t = \pm T_s, \pm 2T_s, \ldots$. The time series corresponding to the special case $\beta = 0$ (the ideal low-pass rectangular spectrum) is $\sin(\pi t/T_s)/(\pi t/T_s)$ just as expected.

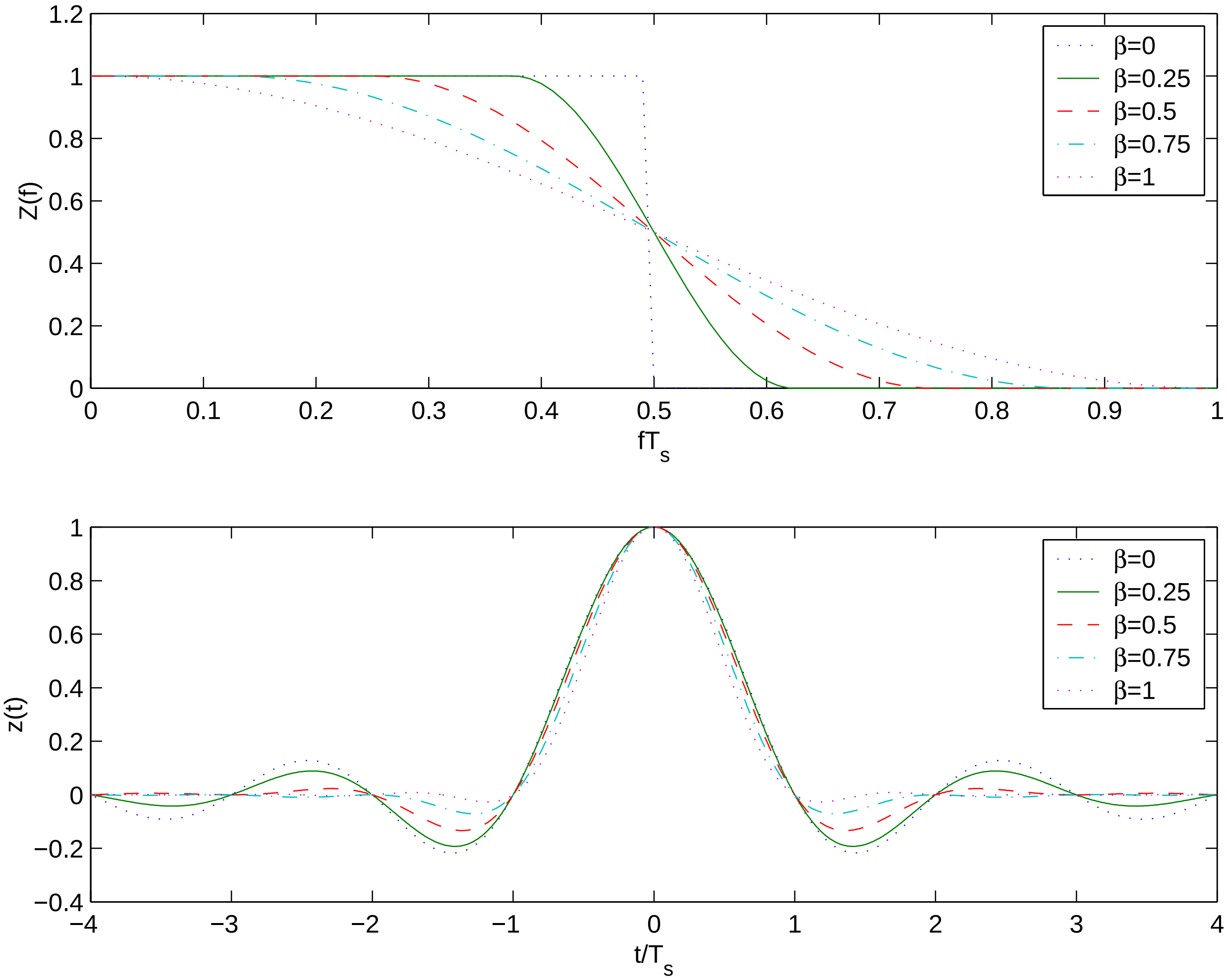

Figure 1: Raised cosine spectra and corresponding time-domain pulses for various values of β.

The spectra and corresponding time series for various values of β are plotted in Figure 1. Note that larger values of β (larger bandwidths) are characterized by a time-domain signal that has faster sidelobe decay rates.

2 Square Root Raised Cosine Spectrum and Pulse Shape

The square-root raised cosine pulse shape $p(t)$ and it's Fourier transform $P(f)$ are given by

$$P(f) = |Z(f)|^{1/2} \tag{4}$$

$$p(t) = \frac{2\beta}{\pi\sqrt{T_s}} \frac{\cos\left[(1+\beta)\pi\frac{t}{T_s}\right] + \frac{\sin\left[(1-\beta)\pi\frac{t}{T_s}\right]}{4\beta\frac{t}{T_s}}}{\left[1 - \left(4\beta\frac{t}{T_s}\right)^2\right]} \tag{5}$$

These functions are plotted in Figure 2. Note that the zero crossings of the time-domain pulse shape are spaced by T_s seconds (i.e. by the symbol time). The spacing between the zero crossings is also a function of the roll-off factor β — as β approaches zero, the spacing approaches T_s.

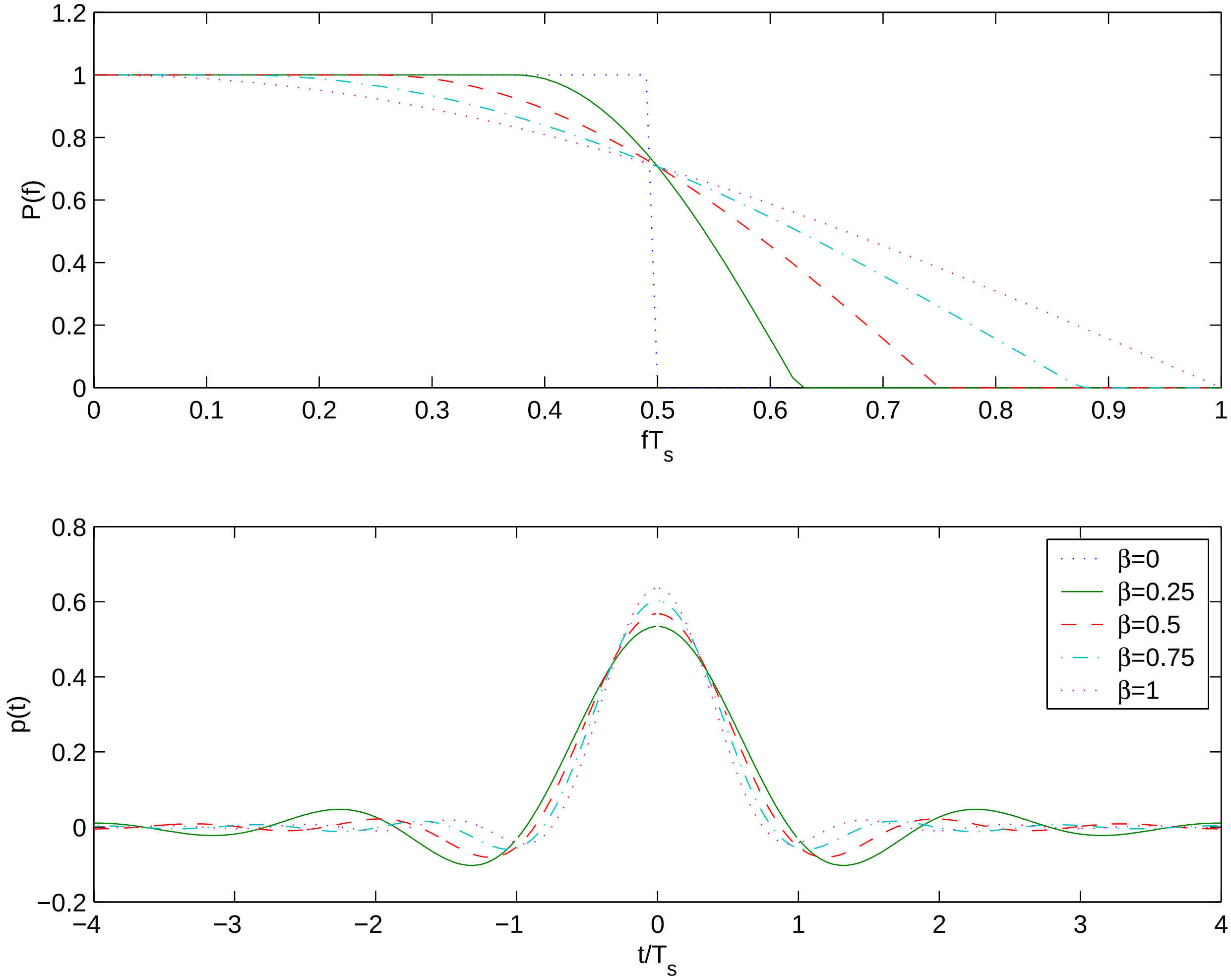

Figure 2: Square-root raised cosine spectra and corresponding time-domain pulses for various values of β.

3 Truncation

The good thing about the square-root raised cosine pulse shape is that the corresponding matched filter output has no ISI. The bad thing is that the pulse shape has infinite support in time. In a practical system, pulses cannot last indefinitely. So the pulse shape is truncated. The result of truncation is the presence of non-zero side lobes in the frequency domain — the spectrum is no longer zero for $|f| > \frac{1+\beta}{2T_s}$. This is illustrated in Figures 3 through 5. In Figure 3, the pulse given by (5) is sampled at $N = 4$ samples/symbol and is truncated to span only 4 symbols as shown in the upper plot. The lower plot of Figure 3 shows the consequence in the frequency domain: high sidelobes and a significant pass-band ripple. The stop band attenuation is only 18 dB which is not enough for practical applications. In Figure 4, the pulse given by (5) is sampled at $N = 4$ samples/symbol and is truncated to span 8 symbols as shown in the upper plot. In the frequency domain, we see that the pass band ripple has been eliminated but and the out-of-band sidelobes are now about 25 dB down. In Figure 5, the pulse given by (5) is sampled at $N = 4$ samples/symbol and is truncated to span 16 symbols as shown in the upper plot. Now the out-of-band sidelobes are about 32 dB down. Clearly, as the time span of the pulse is increased, the spectrum approaches the ideal spectrum.

In general, the smaller the roll-off factor, the longer the pulse shape needs to be in order to achieve a desired stop-band attenuation. Current practice requires a stop band attenuation of about 40 dB. A good rule-of-thumb that achieves this is

$$L_{\text{symbol}} = -44\beta + 33 \tag{6}$$

for $0.2 < \beta \leq< 0.75$ where L_{symbol} is the length of the filter measured in symbols. Clearly, the prediction of $L_{\text{symbol}} = 0$ for $\beta = 0.75$ is an understatement of the required filter length. The values generated by this formula are intended to be starting points. The resulting filter characeteristics should be verified using the DFT.

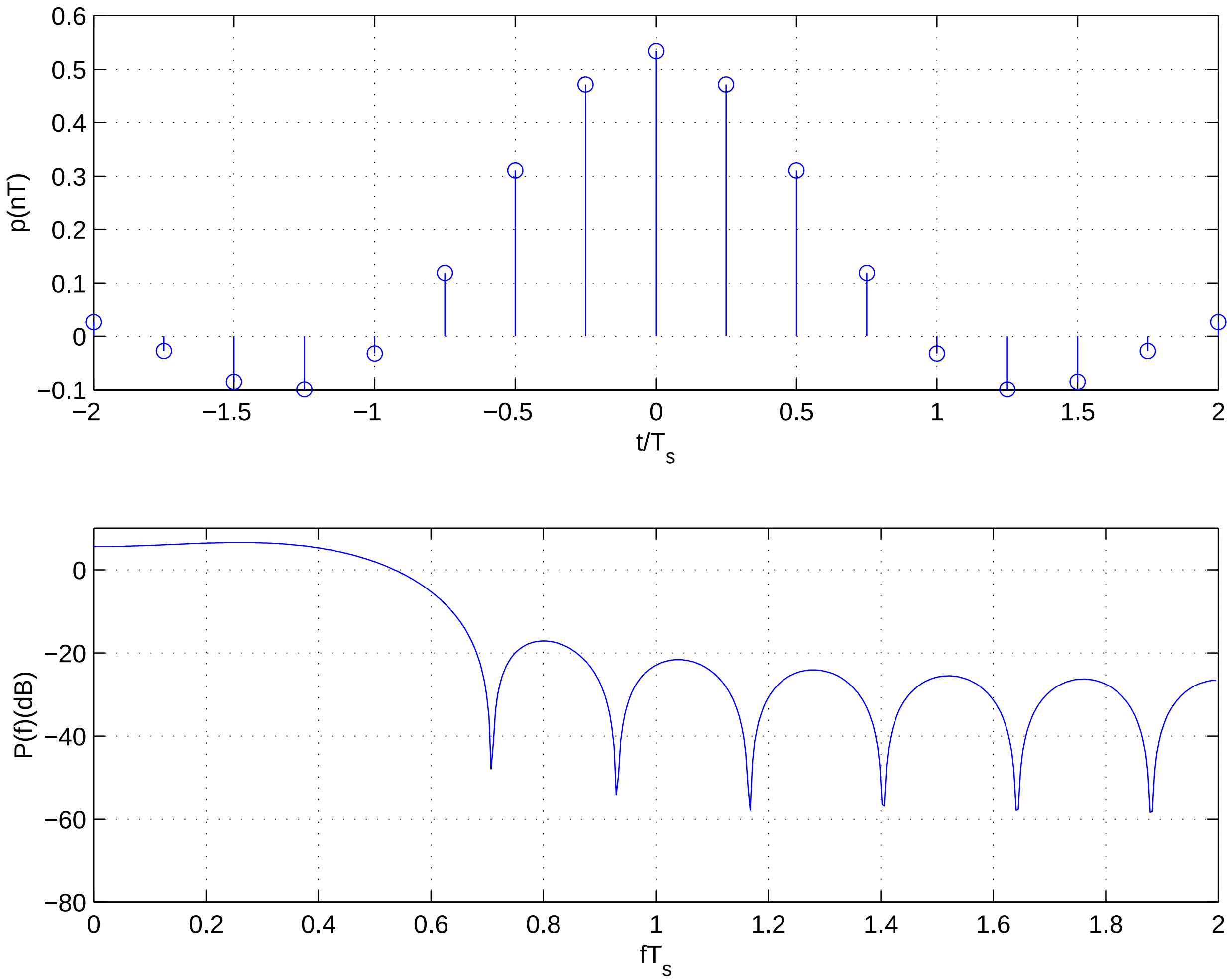

Figure 3: The effects of truncation on the square-root raised cosine pulse shape. Top plot: the square-root raised cosine pulse shape sampled at $N = 4$ samples/symbol with $\beta = 0.5$ and truncated to span 4 symbols. Lower plot: the corresponding spectrum.

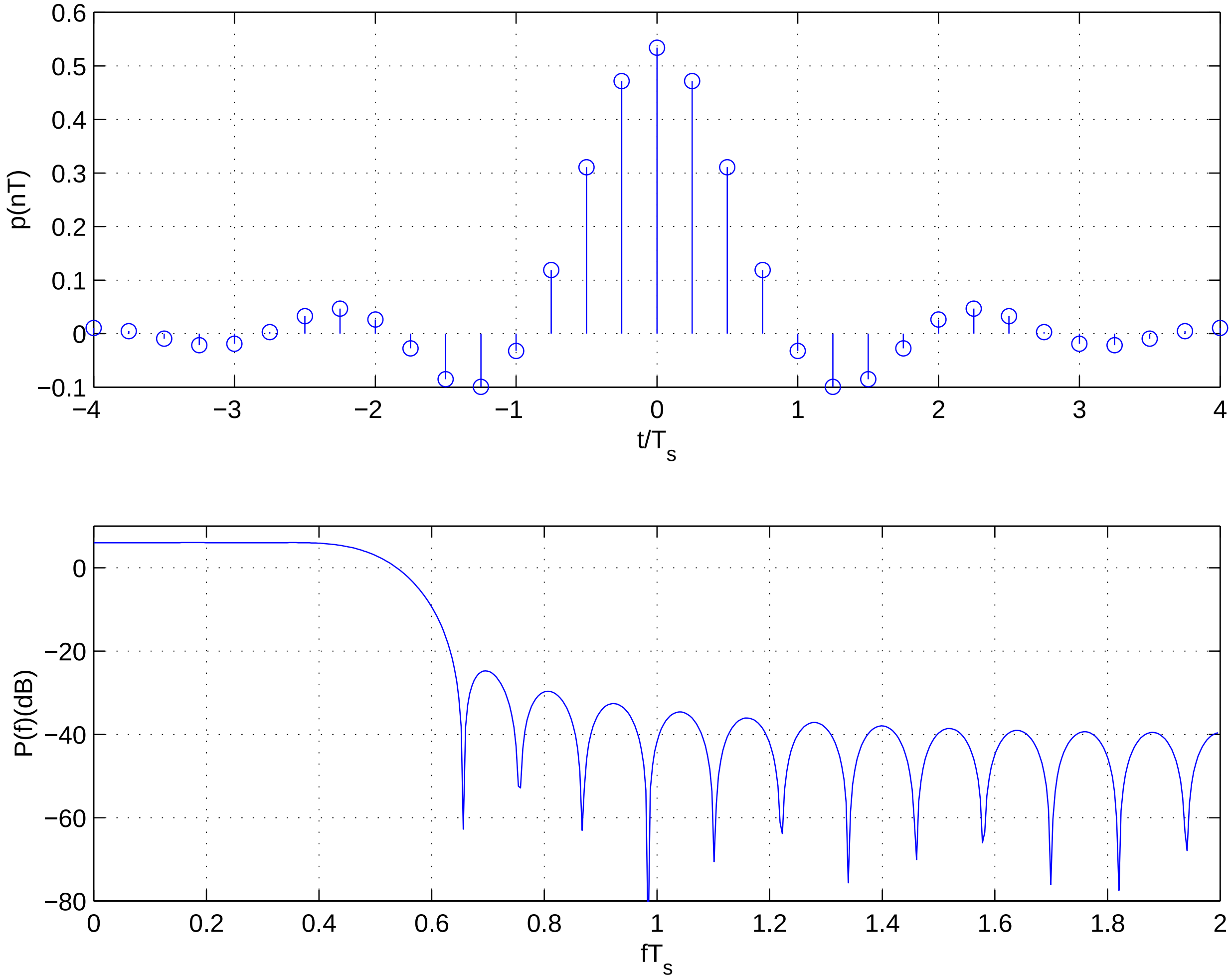

Figure 4: The effects of truncation on the square-root raised cosine pulse shape. Top plot: the square-root raised cosine pulse shape sampled at $N = 4$ samples/symbol with $\beta = 0.5$ and truncated to span 8 symbols. Lower plot: the corresponding spectrum.

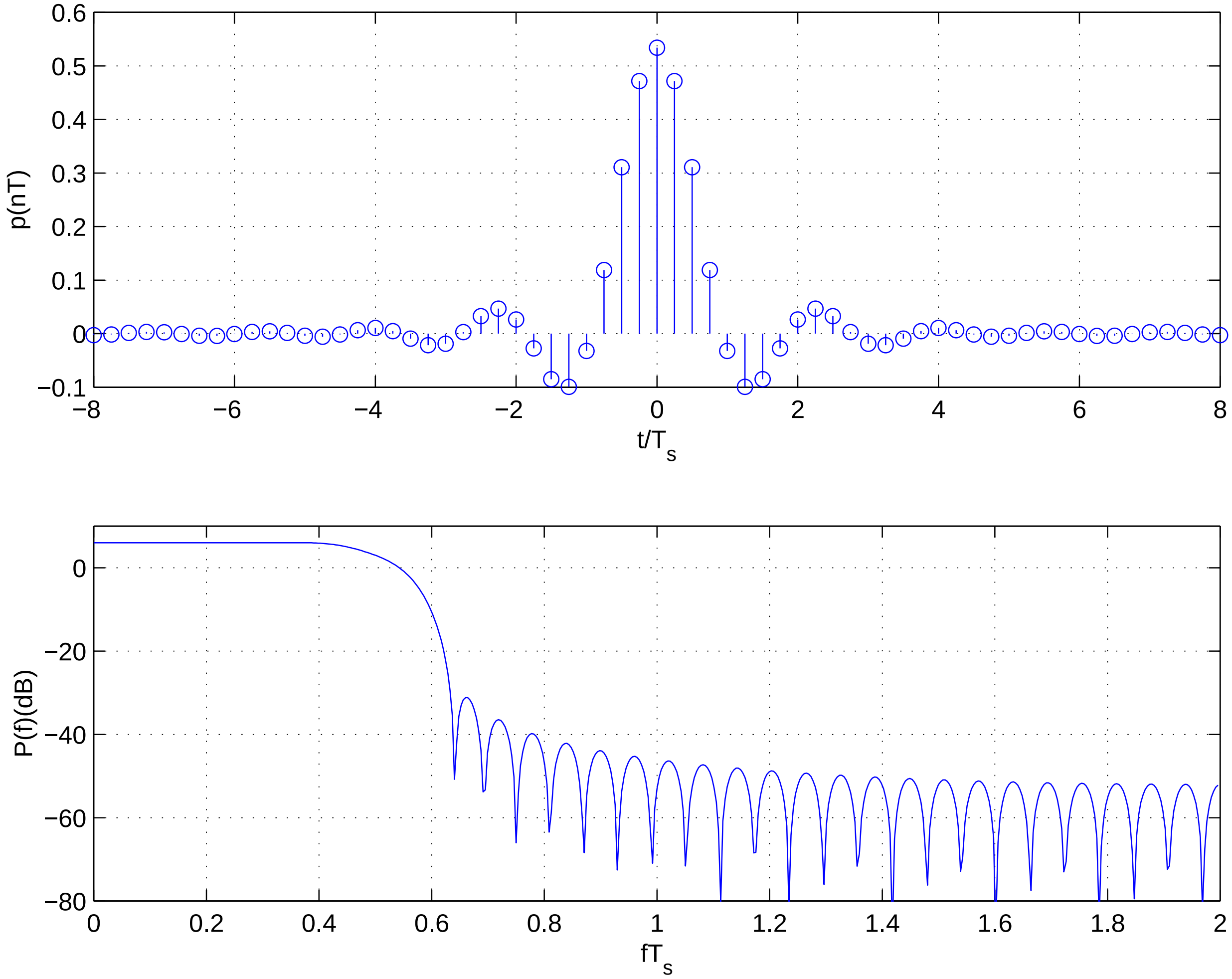

Figure 5: The effects of truncation on the square-root raised cosine pulse shape. Top plot: the square-root raised cosine pulse shape sampled at $N = 4$ samples/symbol with $\beta = 0.5$ and truncated to span 16 symbols. Lower plot: the corresponding spectrum.