Appendix A

Power Spectral Density of Digital Modulation Schemes

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We would like to determine the power spectral density of digital modulation. We are interested both in a general expression of the PSD and in explicit expressions for the standard signal constellations that we have considered. To determine an expression of the PSD of digital modulation, we need to consider not just one signal interval as we have done so far. Instead we consider the input to be an infinite sequence $A[n]$ of symbols. For the explicit expressions for standard signal constellations, we will for simplicity assume that those symbols are independent and identically distributed (i.i.d.).

A.1 Multi-Dimensional Pulse-Amplitude Modulation

Consider a wide sense stationary information process which is an infinite sequence $A[n]$ of symbols from the symbol alphabet used. We will refer to this process as the information process. This sequence is first mapped onto an infinite sequence $S[n]$ of $N$-dimensional signal vectors. Each dimension in this vector sequence is a real valued time-discrete stochastic process $S_i[n]$ of its own, which we will refer to as a component process. Since the information process is wide sense stationary, then the component processes are jointly stationary in the wide sense. These vectors are then modulated using the $N$ basis functions. For the $n$-th vector in that sequence, we use the time-shifted (real-valued) basis functions ${\phi_i(t - nT - \Psi)}_{i=1}^{N}$. Here $\Psi$ is a random delay, independent of $A[n]$ for all $n$, and uniformly distributed over the signalling interval $[0, T)$. This random delay is as usual intended to maintain the wide sense stationarity. The resulting signal is then

$$ S(t) = \sum_n \sum_{i=1}^{N} S_i[n] \phi_i(t - nT - \Psi) $$
i.e. the component processes $S_i[n]$ are pulse-amplitude modulated using their basis functions as pulse shapes and added. Note that we are using the same $\Psi$ in all dimensions to maintain orthogonality among the basis functions.

A.1.1 General expression

We can write the ACF of the resulting signal $S(t)$ as

$$r_S(\tau) = \mathbb{E}\{S(t)S(t + \tau)\}$$

$$= \mathbb{E}\left\{\sum_{n}^{N} \sum_{i=1}^{N} S_i[n] \phi_i(t - nT - \Psi) \sum_{m}^{N} S_i[m] \phi_i(t + \tau - mT - \Psi)\right\}$$

$$= \mathbb{E}\left\{\sum_{n}^{N} \sum_{m}^{N} \sum_{i=1}^{N} S_i[n] S_i[m] \phi_i(t - nT - \Psi) \phi_i(t + \tau - mT - \Psi)\right\}.$$ 

We use the linearity of the expectation to rewrite that as

$$r_S(\tau) = \sum_{n}^{N} \sum_{m}^{N} \sum_{i=1}^{N} \mathbb{E}\{S_i[n] S_i[m]\} \mathbb{E}\{\phi_i(t - nT - \Psi) \phi_i(t + \tau - mT - \Psi)\}.$$ 

From the assumption that the random delay $\Psi$ and the symbol sequence $M[n]$ are independent we find that also the signals $S_i[n]$ are independent of the delay. Then we get

$$r_S(\tau) = \sum_{n}^{N} \sum_{m}^{N} \sum_{i=1}^{N} \sum_{l=1}^{N} \mathbb{E}\{S_i[n] S_i[m]\} \mathbb{E}\{\phi_i(t - nT - \Psi) \phi_l(t + \tau - mT - \Psi)\}.$$ 

We identify the first expectation as the cross correlation of two component processes. Recall that the component processes are jointly stationary in the wide sense. Thus, this cross correlation is a function of $n - m$. The second expectation is the expectation of a function of $\Psi$. Since $\Psi$ is uniformly distributed over $[0, T)$, its probability density is $1/T$ in that interval, and zero elsewhere. These observations give us

$$r_S(\tau) = \sum_{n}^{N} \sum_{m}^{N} \sum_{i=1}^{N} \sum_{l=1}^{N} r_{S_i,S_i}[n - m] \int_{0}^{T} \phi_i(t - nT - \psi) \phi_l(t + \tau - mT - \psi) \frac{1}{T} d\psi$$

We introduce the new variables $k = n - m$ and $u = t - nT - \psi$. Then we get

$$r_S(\tau) = \frac{1}{T} \sum_{k}^{N} \sum_{i=1}^{N} \sum_{l=1}^{N} r_{S_i,S_i}[k] \sum_{n}^{t-nT} \int_{t-nT-T}^{t} \phi_i(u) \phi_l(u + \tau + kT) du$$

$$= \frac{1}{T} \sum_{k}^{N} \sum_{i=1}^{N} \sum_{l=1}^{N} r_{S_i,S_i}[k] \int_{-\infty}^{\infty} \phi_i(u) \phi_l(u + \tau + kT) du$$
Next, the power spectral density of $S(t)$ is the Fourier transform of $r_S(\tau)$. We get

$$R_S(f) = \int_{-\infty}^{\infty} \frac{1}{T} \sum_{k} \sum_{i=1}^{N} \sum_{l=1}^{N} r_{S_i,S_l}[k] \int_{-\infty}^{\infty} \phi_i(u) \phi_l(u + \tau + kT) du \, e^{-j2\pi f \tau} d\tau.$$  

Now, in the inner integral, let $v = u + \tau + kT$. Then we have

$$R_S(f) = \frac{1}{T} \int_{-\infty}^{\infty} \sum_{i=1}^{N} \sum_{l=1}^{N} r_{S_i,S_l}[k] \int_{-\infty}^{\infty} \phi_i(u) \phi_l(v) e^{-j2\pi(v - u - kT)f} du \, dv$$  

$$= \frac{1}{T} \sum_{i=1}^{N} \sum_{l=1}^{N} \sum_{k} r_{S_i,S_l}[k] e^{j2\pi kT} \int_{-\infty}^{\infty} \phi_i(u) e^{j2\pi uf} du \int_{-\infty}^{\infty} \phi_l(v) e^{-j2\pi uf} dv.$$  

We identify the two integrals in the last expressions as Fourier transforms of $\phi_i(-t)$ and $\phi_l(t)$, and the innermost sum as the Fourier transform of $r_{S_i,S_l}[-k]$. I.e. we have

$$R_S(f) = \frac{1}{T} \sum_{i=1}^{N} \sum_{l=1}^{N} R_{S_i,S_l}^{*}[fT] \Phi_i^{*}(f) \Phi_l(f) = \frac{1}{T} \sum_{i=1}^{N} \sum_{l=1}^{N} R_{S_l,S_i}[fT] \Phi_i^{*}(f) \Phi_l(f),$$

where we have used the relation $r_{S_i,S_l}[-k] = r_{S_l,S_i}[k]$, which in the spectral domain corresponds to $R_{S_l,S_i}^{*}[\theta] = R_{S_i,S_l}^{*}[\theta]$. Totally, we can express this formula as the quadratic form

$$R_S(f) = \frac{1}{T} \left( \Phi_1^{*}(f), \ldots, \Phi_N^{*}(f) \right) \left( \begin{array}{cccc} R_{S_1,S_1}[fT] & \cdots & R_{S_N,S_1}[fT] \\ \vdots & \ddots & \vdots \\ R_{S_1,S_N}[fT] & \cdots & R_{S_N,S_N}[fT] \end{array} \right) \left( \begin{array}{c} \Phi_1(f) \\ \vdots \\ \Phi_N(f) \end{array} \right)$$

This expression does not hold only for digital modulation. It is a general expression for simultaneous pulse-amplitude modulation of several stochastic processes using possibly different pulse shapes for each process. We note that the orthonormal property of the basis functions is not at all used above.

We can notice that since the resulting power-spectral density has to be real, then all imaginary parts of the terms $R_{S_l,S_i}[fT] \Phi_i^{*}(f) \Phi_l(f)$ have to cancel out. Actually, they cancel out in pairs, which can be realized by observing that

$$\Phi_i^{*}(f) \Phi_l(f) = \left( \Phi_i^{*}(f) \Phi_l(f) \right)^*$$

holds. As we have noted, we have $R_{S_i,S_l}[\theta] = R_{S_l,S_i}^{*}[\theta]$. Therefore, the imaginary part of the mixed term $R_{S_i,S_l}[fT] \Phi_i^{*}(f) \Phi_l(f)$ cancels out the imaginary part of $R_{S_i,S_l}[fT] \Phi_i^{*}(f) \Phi_l(f)$. This observation will have implications further on.
A.1.2 Special Cases

Independent Symbols

As stated in the introduction, we would like to express the power spectral densities of standard choices of signal constellations and basis functions, for the simple case where the information process $A[n]$ consists of independent identically distributed variables. Therefore, we would like to determine all involved cross spectra $R_{S_i,S_l}[fT]$ for this assumed situation. We should notice that this means that the information process is stationary in the strict sense. The mapping to the vector sequence $\mathbf{S}[n]$ then results in component processes $S_i[n]$, $i \in \{1,2,\ldots,N\}$, that are jointly stationary in the strict sense. Moreover, the independence of the symbols in the information process results in that $S_i[n_1]$ and $S_l[n_2]$ are independent if $n_1 \neq n_2$ holds regardless of $i$ and $l$. This means that the power spectral densities and cross spectra of the component processes become especially simple. Then we have the cross correlation

$$r_{S_i,S_l}[k] = \lambda_{S_i,S_l}[k] + m_{S_i} m_{S_l} = \lambda_{S_i[n],S_l[n]} \delta[k] + m_{S_i} m_{S_l},$$

where $\lambda_{S_i,S_l}[k]$ is the cross covariance of the two component processes and where $\lambda_{S_i[n],S_l[n]}$ is the covariance of the stochastic variables $S_i[n]$ and $S_l[n]$. Consequently, we have the cross spectrum

$$R_{S_i,S_l}[\theta] = \lambda_{S_i[n],S_l[n]} + m_{S_i} m_{S_l} \sum_m \delta(\theta - m).$$

Thus, all we need to do to determine those cross spectra is to analyze the cases $k = 0$ and $k \neq 0$ separately. An important observation that we can make is that these cross spectra are all real valued. We have observed the general relation

$$r_{S_i,S_l}[k] = r_{S_i,S_l}[-k]$$

which gives us

$$R_{S_i,S_l}[\theta] = R_{S_i,S_l}^{\ast}[\theta]$$

But as noted, the imaginary parts are here zero, so we have

$$R_{S_i,S_l}[\theta] = R_{S_i,S_l}[\theta].$$

Uncorrelated dimensions

If $S_i[n + k]$ and $S_l[n]$ are uncorrelated for $i \neq l$ for all $k$, and if $E\{S_i\} = 0$ holds for all $i$, then the general expression of the PSD collapses to

$$R_S(f) = \frac{1}{T} \sum_{i=1}^{N} R_{S_i}[fT] |\Phi_i(f)|^2$$

since all terms in the matrix then are zero except on the main diagonal, where we actually have the power spectral densities of the component processes.
Special basis functions

Our observations that the imaginary parts of the mixed term $R_{S_i,S_i}[fT]\Phi_i^*(f)\Phi_i(f)$ cancels out the imaginary part of $R_{S_i,S_i}[fT]\Phi_i^*(f)\Phi_i(f)$ and that the cross spectra $R_{S_i,S_i}[\theta]$ are real valued have implications. That help us identify an interesting special case.

If $\Phi_i^*(f)\Phi_i(f)$ is purely imaginary for all $i \neq l$ then we have

$$\Phi_i^*(f)\Phi_i(f) = -\Phi_i^*(f)\Phi_i(f),$$

and all mixed terms cancel out. Again, the general expression collapses to

$$R_S(f) = \frac{1}{T} \sum_{i=1}^{N} R_{S_i}[fT] |\Phi_i(f)|^2.$$  

A.1.3 Sine-Shaped Basis Functions

We wish to derive explicit expressions for the power spectral density of standard signal constellations. We have noticed that most of them use sine-shaped basis functions. We are therefore interested in the Fourier transforms of such basis functions.

The same frequency

As usual assume that $2f_0T$ is positive integer, which guarantees that the two basis functions

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_0 t), \quad \text{for } 0 \leq t < T,$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_0 t), \quad \text{for } 0 \leq t < T,$$

are orthonormal. Using standard relations for the Fourier transform, we find that these signals have the Fourier transforms

$$\Phi_1(f) = \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( j^{-2f_0 T} \sin((f + f_0)T) + j^{2f_0 T} \sin((f - f_0)T) \right),$$

$$\Phi_2(f) = j \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( j^{-2f_0 T} \sin((f + f_0)T) - j^{2f_0 T} \sin((f - f_0)T) \right).$$

We can see that we have

$$\Phi_1^*(f)\Phi_2(f) = j \frac{T}{2} \left( \sin^2((f + f_0)T) - \sin^2((f - f_0)T) \right).$$
which obviously is purely imaginary. Thus, for this two-dimensional situation we have the
power spectral density

\[ R_S(f) = \frac{1}{T} \left( R_{S_1}[fT] |\Phi_1(f)|^2 + R_{S_2}[fT] |\Phi_2(f)|^2 \right) \]

according to our observations in Section A.1.2. We have

\[ |\Phi_1(f)|^2 = \frac{T}{2} \left( \text{sinc}((f + f_0)T) + (-1)^2f_0T \text{sinc}((f - f_0)T) \right)^2, \]
\[ |\Phi_2(f)|^2 = \frac{T}{2} \left( \text{sinc}((f + f_0)T) - (-1)^2f_0T \text{sinc}((f - f_0)T) \right)^2. \]

Most often \( f_0 \) is much larger than \( 1/T \), which corresponds to that \( 2f_0T \) is a large integer. Then the mixed terms in the expansion of the squares above will be small compared to the squared terms, and we can use the approximations

\[ |\Phi_1(f)|^2 \approx \frac{T}{2} \left( \text{sinc}^2((f + f_0)T) + \text{sinc}^2((f - f_0)T) \right) \approx |\Phi_2(f)|^2 \]

Based on this observation, we deduce that the bandwidth of a two-dimensional signal con-
stellation using these basis functions can be the same as the bandwidth of a one-dimensional
signal constellation using one of those basis functions. Therefore, it is fairly common to count
pairs of dimensions instead of just dimensions.

**Different frequencies**

For basis functions using different frequencies, the situation is a bit more complicated than
the above, as we will see here. Let \( 2f_1T \) and \( 2f_2T \) be different positive integers, which again
guarantees that the two basis functions

\[ \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_1t), \quad \text{for } 0 \leq t < T, \]
\[ \phi_2(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_2t), \quad \text{for } 0 \leq t < T, \]

are orthonormal. Using standard relations for the Fourier transform, we find that these
signals have the Fourier transforms

\[ \Phi_1(f) = \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( -2f_1T \text{sinc}((f + f_1)T) + j2f_1T \text{sinc}((f - f_1)T) \right), \]
\[ \Phi_2(f) = \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( -2f_2T \text{sinc}((f + f_2)T) + j2f_2T \text{sinc}((f - f_2)T) \right). \]
We obviously have

\[
\Phi_1^*(f)\Phi_2(f) = \frac{T}{2} \left( j^{-2f_1T-2f_2T}\text{sinc}((f + f_1)T)\text{sinc}((f + f_2)T) \\
+ j^{-2f_1T+2f_2T}\text{sinc}((f + f_1)T)\text{sinc}((f - f_2)T) \\
+ j^{2f_1T-2f_2T}\text{sinc}((f - f_1)T)\text{sinc}((f + f_2)T) \\
+ j^{2f_1T+2f_2T}\text{sinc}((f - f_1)T)\text{sinc}((f - f_2)T) \right). 
\]

If both \(2f_1T\) and \(2f_2T\) are even or both are odd, then the exponents \(-2f_1T-2f_2T\), \(-2f_1T+2f_2T\), \(2f_1T-2f_2T\) and \(2f_1T+2f_2T\) are all even. Consequently, all those powers of \(j\) are real, and the complete expression of \(\Phi_1^*(f)\Phi_2(f)\) is real. If instead one of \(2f_1T\) and \(2f_2T\) is odd and the other even, then all those exponents odd, and all those powers of \(j\) are imaginary and the whole expression of \(\Phi_1^*(f)\Phi_2(f)\) is purely imaginary. Thus, in the first case (both even or both odd) we have the general power spectral density expression

\[
R_S(f) = \frac{1}{T} \left( R_{S_1}[fT] |\Phi_1(f)|^2 + 2R_{S_1,S_2}[fT]\text{Re}\{\Phi_1(f)\Phi_2^*(f)\} + R_{S_2}[fT] |\Phi_2(f)|^2 \right)
\]

for this two-dimensional situation. In the second case, though, we have the power spectral density

\[
R_S(f) = \frac{1}{T} \left( R_{S_1}[fT] |\Phi_1(f)|^2 + R_{S_2}[fT] |\Phi_2(f)|^2 \right)
\]

for this two-dimensional situation according to our observations in Section A.1.2. That could also be given directly from the expression in the first case, by observing that \(\text{Re}\{\Phi_1(f)\Phi_2^*(f)\}\) is zero in this second case.

Again, we have the energy spectra

\[
|\Phi_1(f)|^2 = \frac{T}{2} \left( \text{sinc}((f + f_1)T) + (-1)^{2f_1T}\text{sinc}((f - f_1)T) \right)^2,
\]

\[
|\Phi_2(f)|^2 = \frac{T}{2} \left( \text{sinc}((f + f_2)T) + (-1)^{2f_2T}\text{sinc}((f - f_2)T) \right)^2,
\]

of the basis functions, and again we can use the approximations

\[
|\Phi_1(f)|^2 \approx \frac{T}{4} \left( \text{sinc}^2((f + f_1)T) + \text{sinc}^2((f - f_1)T) \right)
\]

\[
|\Phi_1(f)|^2 \approx \frac{T}{4} \left( \text{sinc}^2((f + f_2)T) + \text{sinc}^2((f - f_2)T) \right)
\]

if \(2f_1T\) and \(2f_2T\) are fairly large integers. To keep the total bandwidth of the signal low, we want the two carrier frequencies to be as close to each other as possible, but still chosen such that \(2f_1T\) and \(2f_2T\) are integers to guarantee orthogonality. Thus, we should choose them to fulfill \(2f_1T = 2f_2T \pm 1\).
A.2 One-Dimensional Constellations

For the one-dimensional case, the general expression above simplifies into

\[ R_S(f) = \frac{1}{T} R_{S_1}[fT] |\Phi_1(f)|^2, \]

which we recognize as the expression that holds for pulse-amplitude modulation of a time-discrete stochastic process \( S_1[n] \). This should not be a surprise, since we are using pulse-amplitude modulation in each dimension.

As before, let \( 2f_1T \) be a positive integer. Here it is not needed to guarantee any orthogonality, since we only have one dimension. Instead we simply demand that, in order to make the calculation of the signal energy simple. We have seen that the standard choice of basis function

\[ \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_1 t), \quad \text{for } 0 \leq t < T \]

has Fourier transform

\[ \Phi_1(f) = \sqrt{\frac{T}{2}} e^{-j\pi fT} \left( j^{-2f_1T} \sin((f + f_1)T) + j^{2f_1T} \sin((f - f_1)T) \right), \]

and energy spectrum

\[ |\Phi_1(f)|^2 = \frac{T}{2} \left( \sin((f + f_1)T) + (-1)^{2f_1T} \sin((f - f_1)T) \right)^2. \]

For an information process that consists of independent identically distributed variables, we have seen that the power spectral density of the one component process is

\[ R_{S_1}[\theta] = \sigma_{S_1}^2 + m_{S_1}^2 \sum_m \delta(\theta - m). \]

If \( 2f_1T \) is even, then only two of the impulses in \( R_{S_1}[fT] \) survive the multiplication by \( |\Phi_1(f)|^2 \), namely those at \( \pm f_1 \). All other impulses in \( R_{S_1}[fT] \) coincide with zeros in \( |\Phi_1(f)|^2 \). The resulting power spectral density is then

\[ R_S(f) = \sigma_{S_1}^2 \left( \sin((f + f_1)T) + (-1)^{2f_1T} \sin((f - f_1)T) \right)^2 \]

\[ + \frac{m_{S_1}^2}{2} \left( \delta((f + f_1)T) + \delta((f - f_1)T) \right) \]

If on the other hand \( 2f_1T \) is odd, then no impulses in \( R_{S_1}[fT] \) coincide with zeros in \( |\Phi_1(f)|^2 \). All impulses in \( R_{S_1}[fT] \) survive, but differently scaled. The dominating impulses are those near \( \pm f_1 \).

We note that if \( m_{S_1} \) is zero, then we do not have any of those problems.

In the following, let \( E_{\text{max}} \) and \( E_{\text{avg}} \) denote the largest signal energy and the average signal energy, respectively, in the considered signal constellation. In the average, we assume that all signals are equally probable.
A.2. One-Dimensional Constellations

A.2.1 On-Off Keying

Recall that the vectors used in on-off keying are (0) and $(\sqrt{E_{\text{max}}})$. Assuming independent and equally probable symbols, we get the mean $m_{S_1} = \sqrt{E_{\text{max}}}/2$ and the variance $\sigma^2_{S_1} = E_{\text{max}}/2$. Here we have $E_{\text{max}} = 2E_{\text{avg}}$ The resulting PSD is

$$R_S(f) = \frac{E_{\text{max}}}{8} \left( \text{sinc} \left((f + f_1)T\right) + (-1)^2f_1T \text{sinc} \left((f - f_1)T\right) \right)^2$$

$$+ \frac{E_{\text{max}}}{8} \left( \delta \left((f + f_1)T\right) + \delta \left((f - f_1)T\right) \right)$$

if we plug in that into the one-dimensional expression above. Expressed in the average signal energy, we have

$$R_S(f) = \frac{E_{\text{avg}}}{4} \left( \text{sinc} \left((f + f_1)T\right) + (-1)^2f_1T \text{sinc} \left((f - f_1)T\right) \right)^2$$

$$+ \frac{E_{\text{avg}}}{4} \left( \delta \left((f + f_1)T\right) + \delta \left((f - f_1)T\right) \right).$$

A.2.2 Binary PSK

In binary PSK we have the vectors $(\pm \sqrt{E_{\text{max}}})$. Assuming independent and equally probable symbols, we get the mean $m_{S_1} = 0$ and the variance $E_{\text{max}}$. Here we have $E_{\text{max}} = E_{\text{avg}}$ Plugging that into the one-dimensional expression above, we get

$$R_S(f) = \frac{E_{\text{max}}}{2} \left( \text{sinc} \left((f + f_1)T\right) + (-1)^2f_1T \text{sinc} \left((f - f_1)T\right) \right)^2.$$ 

Expressed in the average signal energy, we have

$$R_S(f) = \frac{E_{\text{avg}}}{2} \left( \text{sinc} \left((f + f_1)T\right) + (-1)^2f_1T \text{sinc} \left((f - f_1)T\right) \right)^2.$$ 

A.2.3 ASK

We have only considered 4-PSK, in which we have the vectors $(\pm \sqrt{E_{\text{max}}})$ and $(\pm \sqrt{E_{\text{max}}}/3)$. Assuming independent and equally probable symbols, we get the mean $m_{S_1} = 0$ and the variance $\frac{2}{9}E_{\text{max}}$. Here we have $E_{\text{max}} = \frac{9}{5} \cdot E_{\text{avg}}$ Plugging that into the one-dimensional expression above, we get

$$R_S(f) = \frac{5E_{\text{max}}}{18} \left( \text{sinc} \left((f + f_1)T\right) + (-1)^2f_1T \text{sinc} \left((f - f_1)T\right) \right)^2.$$ 

Expressed in the average signal energy, we have

$$R_S(f) = \frac{E_{\text{avg}}}{2} \left( \text{sinc} \left((f + f_1)T\right) + (-1)^2f_1T \text{sinc} \left((f - f_1)T\right) \right)^2.$$
A.3 Two-Dimensional Constellations using One Carrier Frequency

For the two-dimensional case with basis functions
\[
\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_0 t), \quad \text{for } 0 \leq t < T,
\]
\[
\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_0 t), \quad \text{for } 0 \leq t < T,
\]
we have seen that the general expression above simplifies into
\[
R_S(f) = \frac{1}{T} \left( R_{S_1}[fT] |\Phi_1(f)|^2 + R_{S_2}[fT] |\Phi_2(f)|^2 \right)
\]
We have seen that these basis functions have Fourier transforms
\[
\Phi_1(f) = \sqrt{\frac{T}{2}} e^{-\pi f T} \left( j^{-2f_0 T} \text{sinc}((f + f_0)T) + j^{2f_0 T} \text{sinc}((f - f_0)T) \right),
\]
\[
\Phi_2(f) = j \sqrt{\frac{T}{2}} e^{-\pi f T} \left( j^{-2f_0 T} \text{sinc}((f + f_0)T) - j^{2f_0 T} \text{sinc}((f - f_0)T) \right)
\]
and energy spectra
\[
|\Phi_1(f)|^2 = \frac{T}{2} \left( \text{sinc}((f + f_0)T) + (-1)^{2f_0 T} \text{sinc}((f - f_0)T) \right)^2,
\]
\[
|\Phi_2(f)|^2 = \frac{T}{2} \left( \text{sinc}((f + f_0)T) - (-1)^{2f_0 T} \text{sinc}((f - f_0)T) \right)^2.
\]
For an information process that consists of independent identically distributed variables, we have seen that the power spectral density of the component processes are
\[
R_{S_1}[\theta] = \sigma_{S_1}^2 + m_{S_1}^2 \sum_m \delta(\theta - m),
\]
\[
R_{S_2}[\theta] = \sigma_{S_2}^2 + m_{S_2}^2 \sum_m \delta(\theta - m).
\]
The situation is the same as for the one-dimensional case when it comes to even and odd \(2f_1 T\). If \(2f_1 T\) is even, then the resulting power spectral density is
\[
R_S(f) = \frac{\sigma_{S_1}^2}{2} \left( \text{sinc}((f + f_0)T) + (-1)^{2f_0 T} \text{sinc}((f - f_0)T) \right)^2
\]
\[
+ \frac{m_{S_1}^2}{2} \left( \delta((f + f_0)T) + \delta((f - f_0)T) \right)
\]
\[
+ \frac{\sigma_{S_2}^2}{2} \left( \text{sinc}((f + f_0)T) - (-1)^{2f_0 T} \text{sinc}((f - f_0)T) \right)^2
\]
\[
+ \frac{m_{S_2}^2}{2} \left( \delta((f + f_0)T) + \delta((f - f_0)T) \right)
\]
A.3. Two-Dimensional Constellations using One Carrier Frequency.

Again, if \(2f_1T\) is odd, then no impulses in coincide with zeros in \(|\Phi_1(f)|^2\). All impulses in \(R_{S_1}[fT]\) survive, but differently scaled. The dominating impulses are those near \(\pm f_0\).

We note that if \(m_{S_1}\) and \(m_{S_2}\) are zero, then we do not have any of those problems.

A.3.1 Non-Binary PSK

For \(M\)-ary PSK, we have \(M\) signal points evenly distributed over a circle of radius \(\sqrt{E}\) with its center in the origin, where \(E\) is both the largest and average energy. Both component processes have mean zero, and variance \(E/2\). The resulting power spectral density is

\[
R_S(f) = \frac{E}{4} \left( \text{sinc}\left((f + f_0)T\right) + (-1)^2f_0T\text{sinc}\left((f - f_0)T\right) \right)^2 + \frac{E}{4} \left( \text{sinc}\left((f + f_0)T\right) - (-1)^2f_0T\text{sinc}\left((f - f_0)T\right) \right)^2
\]

\[
= \frac{E}{2} \left( \text{sinc}^2((f + f_0)T) + \text{sinc}^2((f - f_0)T) \right)
\]

A.3.2 QAM

We have considered 16-QAM in detail. There we have 16 points symmetrically placed on a rectangular grid. In each dimension, we have 4-ASK. Both component processes have mean zero, and variance \(5E_{\text{max}}/18\). The resulting power spectral density is

\[
R_S(f) = \frac{5E_{\text{max}}}{36} \left( \text{sinc}\left((f + f_0)T\right) + (-1)^2f_0T\text{sinc}\left((f - f_0)T\right) \right)^2 + \frac{5E_{\text{max}}}{36} \left( \text{sinc}\left((f + f_0)T\right) - (-1)^2f_0T\text{sinc}\left((f - f_0)T\right) \right)^2
\]

\[
= \frac{5E_{\text{max}}}{18} \left( \text{sinc}^2((f + f_0)T) + \text{sinc}^2((f - f_0)T) \right)
\]

As for 4-ASK, we have \(E_{\text{max}} = \frac{9}{5} \cdot E_{\text{avg}}\). Expressed in the average signal energy, we have

\[
R_S(f) = \frac{E_{\text{avg}}}{2} \left( \text{sinc}\left((f + f_1)T\right) + (-1)^2f_1T\text{sinc}\left((f - f_1)T\right) \right)^2
\]