Information Theory for Wireless Communications, Part II:

Lecture 5: Multiuser Gaussian MIMO Multiple-Access Channel

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In this lecture, we give the capacity region of the Gaussian multiuser multiple-input multiple-output (MIMO) multiple-access channel (MAC). We also give an algorithm for computing the maximum sum-capacity of the Gaussian MIMO MAC for the 2-user scenario.

I. General System Model

The system under study is illustrated in Fig. 1. There are K multiple-antenna users that want to transmit data to a single receiver equipped with multiple antennas. User $k \in \{1,\ldots,K\}$ has n_k antennas and the receiver has n_r antennas. The channel between user k and the receiver is $\boldsymbol{H}_k \in \mathbb{C}^{n_r \times n_k}$. User k transmits a vector $\boldsymbol{x}_k \in \mathbb{C}^{n_k}$. The transmissions are concurrent and take place in the same band, so the signals will sum up at the receiver. Hence, the received signal is

$$\boldsymbol{y} = \sum_{k=1}^{K} \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{z},$$

where $\boldsymbol{z} \in \mathbb{C}^{n_r}$ is the additive noise that we model as a zero-mean complex-symmetric Gaussian random vector with covariance $N_0\boldsymbol{I}$. The transmitted vectors are subject to a power constraint

$$\mathrm{Tr}\big\{\mathrm{E}\big\{\boldsymbol{x}_k\boldsymbol{x}_k^H\big\}\big\} = \mathrm{E}\big\{\|\boldsymbol{x}_k\|^2\big\} \le P_k.$$

II. Review of the Capacity Region for the SISO Two-User MAC

We studied the two-user SISO MAC in [1] and [2]. Here we revisit a few of these results. Consider the input distribution $p(x_1,x_2) = p(x_1)p(x_2)$. This input distribution corresponds to independent encoding. For a given input distribution, we can achieve the rate region

$$\begin{aligned} R_1 &\le I(X_1;Y|X_2),\\ R_2 &\le I(X_2;Y|X_1),\\ R_1 + R_2 &\le I(X_1,X_2;Y). \end{aligned}$$

An example of this region is illustrated in Fig. 2.

Note that

$$I(X_1,X_2;Y) = I(X_1;Y|X_2) + I(X_2;Y) = I(X_2;Y|X_1) + I(X_1;Y).$$

The term $I(X_1;Y)$ is the mutual information we get when X_1 is decoded treating X_2 as noise (interference). The point A in Fig. 2 is the rate pair we achieve when we first decode user 1 and then user 2.

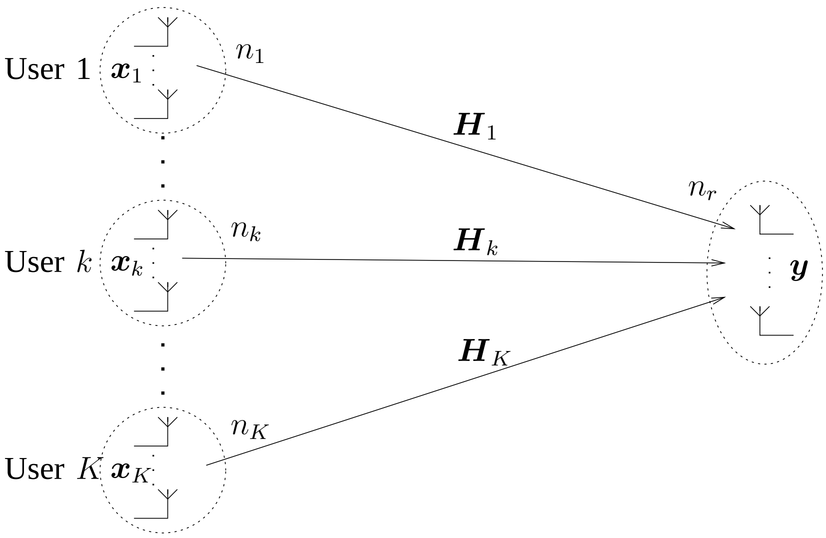

Fig. 1. System model of the Gaussian multi-user MIMO MAC.

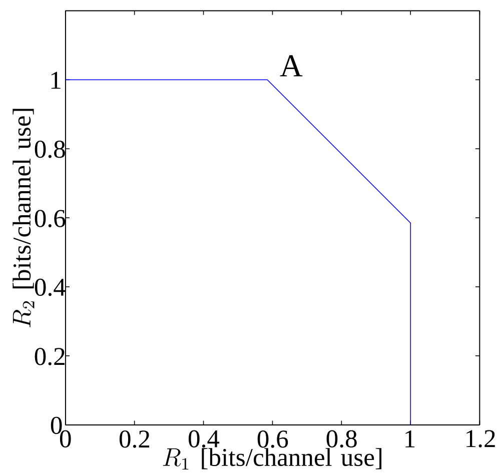

Fig. 2. Achievable rate region of the SISO MAC for a given input distribution $p(x_1, x_2) = p(x_1)p(x_2)$.

For the Gaussian case, we have the (simplified) signal model

$$Y = X_1 + X_2 + Z$$

with $\mathrm{E}\{X_k\} \leq P_k$ and $Z \sim \mathcal{CN}(0, N_0)$. The best rate of user 1 in the absence of user 2 is

$$R_1 \leq \log_2\left(1 + \frac{P_1}{N_0}\right).$$

In the same way, the best rate of user 2 is

$$R_2 \leq \log_2\left(1 + \frac{P_2}{N_0}\right).$$

Consider the case where the receiver first decodes user 2 treating X_1 as noise. The new signal model is then $Y = X_2 + Z'$ with $Z' \triangleq X_1 + Z$. If $X_1 \sim \mathcal{CN}(0, P_1)$ then $Z' \sim \mathcal{CN}(0, P_1 + N_0)$ and user 2 can achieve rate

$$R_2' \le \log_2\left(1+\frac{P_2}{P_1+N_0}\right).$$

When X_2 is decoded, the receiver subtracts it from the received signal and obtains the signal

$$Y' \triangleq Y - X_2 = X_1 + Z.$$

Now, user 1 can achieve rate

$$R_1' \le I(X_1;Y') = \log_2\left(1+\frac{P_1}{N_0}\right).$$

Note that $R_1' + R_2' \le \log_2(1+(P_1+P_2)/N_0)$.

III. Capacity Region for Gaussian SIMO MAC

Here, we consider the scenario of where $n_k = 1, k = 1, \ldots, K$ and $n_r > 1$. This setup is the single-input multiple-output (SIMO) MAC. For the two-user case, the signal model for the SIMO MAC is

$$\boldsymbol{Y} = \boldsymbol{h}_1 X_1 + \boldsymbol{h}_2 X_2 + Z.$$

Let us assume that $\mathrm{E}\{|X_k|^2\} = 1$. Define $\boldsymbol{H} = [\boldsymbol{h}_1 \;\; \boldsymbol{h}_2]$. The capacity region rate region of this system is given by

$$R_1 \le I(X_1;\boldsymbol{Y}|X_2) = \log_2\left(1+\frac{\|\boldsymbol{h}_1\|^2 P_1}{N_0}\right) \tag{1}$$

$$R_2 \le I(X_2;\boldsymbol{Y}|X_1) = \log_2\left(1+\frac{\|\boldsymbol{h}_2\|^2 P_2}{N_0}\right) \tag{2}$$

$$R_1 + R_2 \le I(X_1, X_2;\boldsymbol{Y}) = \max_{0\le\alpha_k\le P_k} \log_2\left|\boldsymbol{I} + \frac{1}{N_0}\boldsymbol{H}\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}\boldsymbol{H}^H\right|. \tag{3}$$

The rates (1) and (2) are the single-user rates. To see how we get to (3), we consider a point-to-point (single-user) MIMO link

$$\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{X} + \boldsymbol{Z}$$

with $\boldsymbol{X} = [X_1, X_2]^T$, $\boldsymbol{K}_X \triangleq \mathrm{E}\{\boldsymbol{X}\boldsymbol{X}^H\}$, and $\boldsymbol{Z} \sim \mathcal{CN}(\boldsymbol{0}, N_0 I)$. The capacity of such channel is

$$\max_{\mathrm{Tr}\{\boldsymbol{K}_X\}\le P, \boldsymbol{K}_X \succeq \boldsymbol{0}} \log_2\left|\boldsymbol{I} + \frac{1}{N_0}\boldsymbol{H}\boldsymbol{K}_X\boldsymbol{H}^H\right|.$$

In our case, we restrict to the case where the encoding is independent, i.e., the signals at the transmit antennas are independent. This restriction gives us a diagonal covariance matrix $\boldsymbol{K}_X$. Also, we have a per antenna power constraint. Next we will show that in order to maximize (3), we must have $\alpha_k = P_k$. Due to the monotonicity of the logarithm, maximize (3) is equivalent to maximizing

$$\begin{aligned}
&\left|\boldsymbol{I} + \frac{1}{N_0}\boldsymbol{H}\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}\boldsymbol{H}^H\right| = \left|\boldsymbol{I} + \frac{1}{N_0}[\alpha_1\boldsymbol{h}_1 \;\; \alpha_2\boldsymbol{h}_2]\begin{bmatrix}\boldsymbol{h}_1^H\\ \boldsymbol{h}_2^H\end{bmatrix}\right| = \Big\{|\boldsymbol{I}+\boldsymbol{A}\boldsymbol{B}| = |\boldsymbol{I}+\boldsymbol{B}\boldsymbol{A}|\Big\}\\
&= \left|\boldsymbol{I} + \frac{1}{N_0}\begin{bmatrix}\boldsymbol{h}_1^H\\ \boldsymbol{h}_2^H\end{bmatrix}[\alpha_1\boldsymbol{h}_1 \;\; \alpha_2\boldsymbol{h}_2]\right| = \left|\boldsymbol{I} + \frac{1}{N_0}\begin{bmatrix}\alpha_1\|\boldsymbol{h}_1\|^2 & \alpha_2\boldsymbol{h}_1^H\boldsymbol{h}_2\\ \alpha_1\boldsymbol{h}_2^H\boldsymbol{h}_1 & \alpha_2\|\boldsymbol{h}_2\|^2\end{bmatrix}\right|\\
&= \log_2\left(\left(1+\frac{\alpha_1}{N_0}\|\boldsymbol{h}_1\|^2\right)\left(1+\frac{\alpha_2}{N_0}\|\boldsymbol{h}_2\|^2\right) - \frac{\alpha_1\alpha_2}{N_0}|\boldsymbol{h}_1^H\boldsymbol{h}_2|^2\right)\\
&= \log_2\left(1+\frac{\alpha_1}{N_0}\|\boldsymbol{h}_1\|^2 + \frac{\alpha_2}{N_0}\|\boldsymbol{h}_2\|^2 + \frac{\alpha_1\alpha_2}{N_0}\left(\|\boldsymbol{h}_1\|^2\|\boldsymbol{h}_2\|^2 - |\boldsymbol{h}_1^H\boldsymbol{h}_2|^2\right)\right).
\end{aligned} \tag{4}$$

Since $\|\boldsymbol{h}_1\|^2 \|\boldsymbol{h}_2\|^2 - |\boldsymbol{h}_1^H \boldsymbol{h}_2|^2 \geq 0$ (equality only if $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are colinear), we maximize (4) for $\alpha_k^\star = P_k$, $k = 1, 2$. Hence, we can write (3) as

$$R_1 + R_2 \leq \log_2 \left| \boldsymbol{I} + \frac{1}{N_0} \boldsymbol{H} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \boldsymbol{H}^H \right|.$$

It is clear that also this capacity region is a pentagon similar to that in Fig. 2. At the corner point A, we have the rate pair

$$R_1 = \log_2(1 + P_1 \boldsymbol{h}_1^H (N_0 \boldsymbol{I} + P_2 \boldsymbol{h}_2 \boldsymbol{h}_2^H)^{-1} \boldsymbol{h}_1) \tag{5}$$

$$R_2 = \log_2 \left(1 + \frac{P_2 \|\boldsymbol{h}_2\|^2}{N_0} \right). \tag{6}$$

We achieve this point by first decoding user 1, treating user 2 as noise. The components of the noise vector $\boldsymbol{h}_2 X_2 + \boldsymbol{Z}$ is spatially correlated with covariance $P_2 \boldsymbol{h}_2 \boldsymbol{h}_2^H + N_0 \boldsymbol{I}$. So, we prewhiten the received signal, i.e.,

$$\tilde{\boldsymbol{Y}} = (\boldsymbol{h}_2 \boldsymbol{h}_2^H + N_0 \boldsymbol{I})^{-1/2} \boldsymbol{Y} = \underbrace{(\boldsymbol{h}_2 \boldsymbol{h}_2^H + N_0 \boldsymbol{I})^{-1/2} \boldsymbol{h}_1}_{\triangleq \tilde{\boldsymbol{h}}_1} X_1 + \underbrace{(\boldsymbol{h}_2 \boldsymbol{h}_2^H + N_0 \boldsymbol{I})^{-1/2} (\boldsymbol{h}_2 X_2 + \boldsymbol{Z})}_{\triangleq \tilde{\boldsymbol{Z}}}.$$

Since $\tilde{\boldsymbol{Z}} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I})$, the rate of user 1 is $\log_2 \left(1 + P_1 \left\| \tilde{\boldsymbol{h}}_1 \right\|^2 \right) = \log_2(1 + P_1 \boldsymbol{h}_1^H (N_0 \boldsymbol{I} + P_2 \boldsymbol{h}_2 \boldsymbol{h}_2^H)^{-1} \boldsymbol{h}_1)$. Once user 1 is decoded, the term is subtracted from the received signal and user 2 is decoded, and it can achieve the rate $R_2 = \log_2 \left(1 + P_2 \|\boldsymbol{h}_2\|^2 / N_0 \right)$.

A. Degrees of Freedom

We study the behavior of the rates (5) and (6) when $P_1, P_2 \to \infty$. First, we note that

$$\lim_{P_2 \to \infty} \frac{R_2}{\log_2 P_2} = 1.$$

Second, we have

$$\begin{aligned} \lim_{P_1, P_2 \to \infty} \frac{R_1}{\log_2 P_1} &= \left\{ \text{Matrix inversion lemma} \right\} \\ &= \lim_{P_1, P_2 \to \infty} \frac{1}{\log_2 P_1} \log_2 \left(1 + \frac{P_1}{N_0} \left(\|\boldsymbol{h}_1\|^2 - \frac{|\boldsymbol{h}_1^H \boldsymbol{h}_2|^2}{N_0/P_2 + \|\boldsymbol{h}_2\|^2} \right) \right) \\ &= \left\{ P_2 \to \infty \right\} = \lim_{P_1 \to \infty} \frac{1}{\log_2 P_1} \log_2 \left(1 + \frac{P_1}{N_0} \left(\|\boldsymbol{h}_1\|^2 - \frac{|\boldsymbol{h}_1^H \boldsymbol{h}_2|^2}{\|\boldsymbol{h}_2\|^2} \right) \right) = 1 \end{aligned}$$

To conclude, for the two-user SIMO MAC, we achieve two degrees of freedom. This is a gain over the SISO MAC, where we achieved only a single degree of freedom.

B. Orthogonal Transmission Scheme

Consider the scenario where the transmissions of users 1 and 2 are divided in time (or frequency) in such way that they do not interfere with each other at the receiver. That is, user 1 uses the channel for a fraction α of the time. During that time it transmits using power P_1/α in order to use power P_1 in average. For the remaining fraction $1 - \alpha$ of the time, user 2 transmits using power $P_2/(1 - \alpha)$. For this scheme, the achievable rates are

$$R_1 = \alpha \log_2 \left(1 + \frac{P_1}{\alpha N_0} \right) \qquad R_2 = (1 - \alpha) \log_2 \left(1 + \frac{P_2}{(1 - \alpha) N_0} \right)$$

for the SISO MAC and

$$R_1 = \alpha \log_2\left(1 + \frac{P_1 \|\boldsymbol{h}_1\|^2}{\alpha N_0}\right) \qquad R_2 = (1-\alpha)\log_2\left(1 + \frac{P_2 \|\boldsymbol{h}_2\|^2}{(1-\alpha) N_0}\right)$$

for the SIMO MAC.By varying α from 0 to 1, we get the boundaries of the achievable rate regions. As illustrated in Fig. 3, we see that the orthogonal transmission scheme is suboptimal. However, for the SISO MAC, the orthogonal scheme achieves the sum-capacity [3, Exercise 10.4].

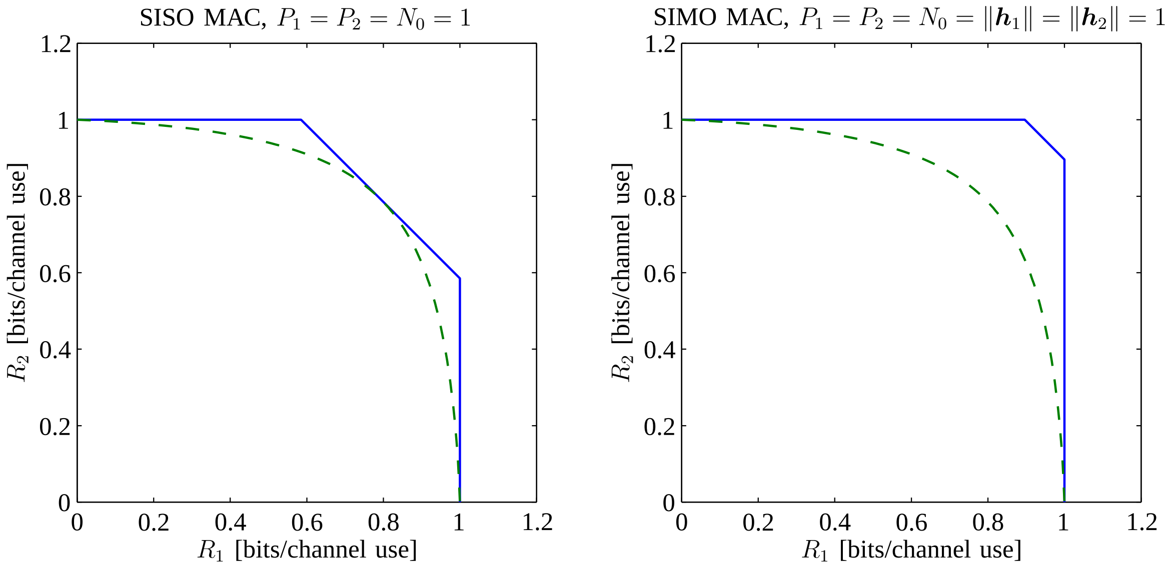

Fig. 3. Illustration of the optimality of orthogonal transmission schemes. The solid line is the boundary of the capacity region whereas the dashed line is the boundary of the region that is achievable by orthogonal transmission.

IV. Two-User Gaussian MIMO-MAC

Now, we extend the model to the scenario of multiple antennas at transmitters and receiver. We focus on the two-user case. For a given pair of transmit covariance matrices $(\boldsymbol{K}_1, \boldsymbol{K}_2)$, we have the achievable rate region

$$R_1 \le \log_2\left|\boldsymbol{I} + \frac{1}{N_0}\boldsymbol{H}_1\boldsymbol{K}_1\boldsymbol{H}_1^H\right|, \tag{7}$$

$$R_2 \le \log_2\left|\boldsymbol{I} + \frac{1}{N_0}\boldsymbol{H}_2\boldsymbol{K}_2\boldsymbol{H}_2^H\right|, \tag{8}$$

$$R_1 + R_2 \le \log_2\left|\boldsymbol{I} + \frac{1}{N_0}\begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{H}_1^H \\ \boldsymbol{H}_2^H \end{bmatrix}\right|. \tag{9}$$

We also have the power constraints $\mathrm{Tr}\{\boldsymbol{K}_k\} \le P_k,\ k = 1, 2$. The reason for why the joint covariance matrix in (9) is block-diagonal is that the encoding of the users' messages is independent.

Note that the pair of covariance matrices $(\boldsymbol{K}_1, \boldsymbol{K}_2)$ that maximizes (7) and (8) does not maximize (9) in general. To find the capacity region, we have to take the union over all pair of feasible transmit covariance matrices. This also implies that the capacity region of the Gaussian MIMO MAC is not a pentagon.

Next, we will focus on the sum-capacity and we will introduce the so-called iterative water-filling algorithm to solve

$$\max_{\mathrm{Tr}\{\boldsymbol{K}_1\}\le P_k, \boldsymbol{K}_k \succeq \boldsymbol{0}} \log_2 \left| \boldsymbol{I} + \frac{1}{N_0}\left(\boldsymbol{H}_1\boldsymbol{K}_1\boldsymbol{H}_1^H + \boldsymbol{H}_2\boldsymbol{K}_2\boldsymbol{H}_2^H\right)\right|.$$

In the remainder of this section, we assume that $N_0 = 1$.

First, we start with a feasible pair of transmit covariance matrices $(\boldsymbol{K}_1^{(0)}, \boldsymbol{K}_2^{(0)})$ for which we achieve the rate

$$C(\boldsymbol{K}_1^{(0)}, \boldsymbol{K}_2^{(0)}) = \log_2 \left| \boldsymbol{H}_1\boldsymbol{K}_1^{(0)}\boldsymbol{H}_1^H + (\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{K}_2^{(0)}\boldsymbol{H}_2^H)\right|. \tag{10}$$

Compare this to the point-to-point scenario $\boldsymbol{Y} = \boldsymbol{H}_1\boldsymbol{X}_1 + \boldsymbol{W}$, where the noise vector $\boldsymbol{W}$ has the covariance $\boldsymbol{K}_{\boldsymbol{W}}^{(0)} \triangleq \boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{K}_2^{(0)}\boldsymbol{H}_2^H$. The mutual information is then $\log_2\left|\boldsymbol{K}_{\boldsymbol{W}}^{(0)} + \boldsymbol{H}_1\boldsymbol{K}_1\boldsymbol{H}_1^H\right| - \log_2\left|\boldsymbol{K}_{\boldsymbol{W}}^{(0)}\right|$. Hence, we can write (10) as

$$C(\boldsymbol{K}_1^{(0)}, \boldsymbol{K}_2^{(0)}) = I(\boldsymbol{K}_1 = \boldsymbol{K}_1^{(0)}) + \log_2\left|\boldsymbol{K}_{\boldsymbol{W}}^{(0)}\right| \le I(\boldsymbol{K}_1 = \boldsymbol{K}_1^{(1)}) + \log_2\left|\boldsymbol{K}_{\boldsymbol{W}}^{(0)}\right|$$

where

$$\boldsymbol{K}_1^{(1)} = \operatorname*{argmax}_{\mathrm{Tr}\{\boldsymbol{K}_1\}\le P_1, \boldsymbol{K}_1 \succeq \boldsymbol{0}} I_{\boldsymbol{K}_{\boldsymbol{W}}^{(0)}}(\boldsymbol{X}_1; \boldsymbol{Y}) \neq \boldsymbol{K}_1^{(0)}.$$

We solve this using the same water-filling algorithm as we use for the point-to-point MIMO channel. Now, for the pair $(\boldsymbol{K}_1^{(1)}, \boldsymbol{K}_2^{(0)})$, we have the capacity

$$C(\boldsymbol{K}_1^{(1)}, \boldsymbol{K}_2^{(0)}) = \log_2 \left| \boldsymbol{H}_2\boldsymbol{K}_2^{(0)}\boldsymbol{H}_2^H + (\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{K}_1^{(1)}\boldsymbol{H}_1^H)\right|.$$

Next, we maximize with respect to $\boldsymbol{K}_2$, and so on until convergence. We see that we have an improvement of the sum-rate in each iteration. Since the sum-capacity is bounded, the iterative water-filling algorithm. A pair $(\boldsymbol{K}_1^\star, \boldsymbol{K}_2^\star)$ is a optimal solution if

$$\boldsymbol{K}_1^\star = \operatorname*{argmax}_{\mathrm{Tr}\{\boldsymbol{K}_1\}\le P_1, \boldsymbol{K}_1 \succeq \boldsymbol{0}} \log_2 \left| \boldsymbol{H}_1\boldsymbol{K}_1\boldsymbol{H}_1^H + (\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{K}_2^\star\boldsymbol{H}_2^H)\right|$$

and vice versa for $\boldsymbol{K}_2^\star$.

In Fig. 4, we illustrate the capacity region for the two-user Gaussian MIMO MAC with

$$\boldsymbol{H}_1 = \begin{bmatrix} 1.9583 - 0.0446i & 2.1460 - 0.1449i \\ -0.9545 + 0.5054i & 0.5129 - 0.0878i \end{bmatrix}, \quad \boldsymbol{H}_2 = \begin{bmatrix} 1.0534 - 0.8538i & 1.0021 + 1.1528i \\ 0.9963 + 0.5072i & 0.4748 + 0.3457i \end{bmatrix},$$

and $P_1 = P_2 = N_0 = 1$. As we can see, the region does not have the shape of a pentagon.

References

[1] H. Q. Ngo, "Information Theory for Wireless Communications: Part I, Lecture 12: Single-antenna multi-user uplink channels (achievable rate region)."
[2] "Information Theory for Wireless Communications: Part I, Lecture 13: Single-antenna multi-user uplink channels (capacity region and converse)."
[3] D. Tse and P. Viswanath, "Fundamentals of Wireless Communication," Cambridge University Press, 2008.

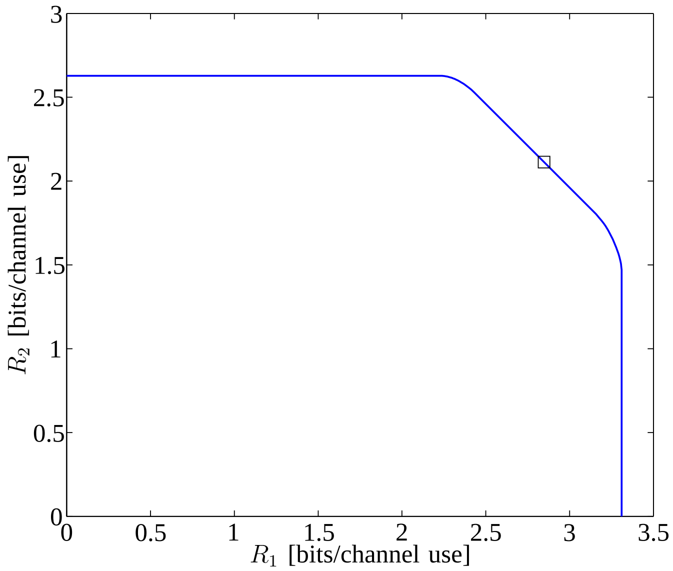

Fig. 4. Illustration of the two-user Gaussian MIMO MAC capacity region. The square marks the sum-capacity point.