ITWC PART - II

HOMEWORK - I

SUBMISSION DATE: OCT 8, 2012

Maximum points : 100.
passing points : 60.

Q.1 (2 user) DM-BC with common message.

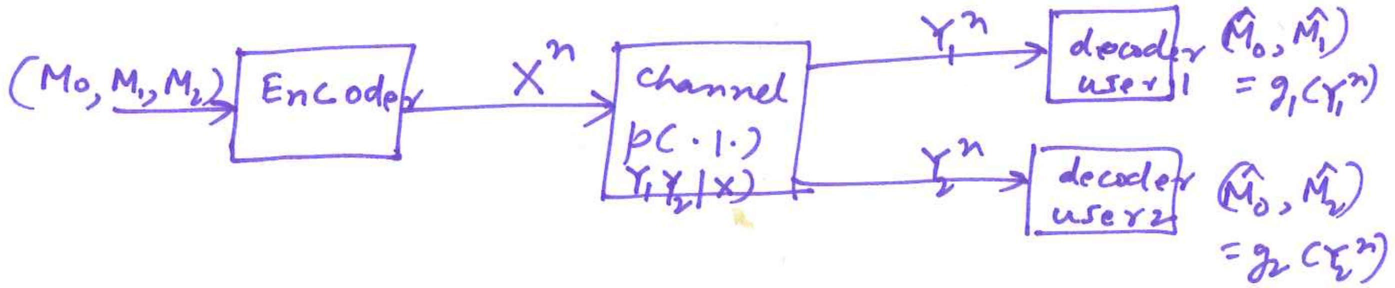
(25 points)

Messages for user 1 : M_0, M_1

Messages for user 2 : M_0, M_2.

Common message : M_0.

$\mathcal{M}_0 = \{1 \dots 2^{nR_0}\}$, $\mathcal{M}_1 = \{1, \dots, 2^{nR_1}\}$, $\mathcal{M}_2 = \{1 \dots 2^{nR_2}\}$.



Prob of error for first user $P_{e_1}^{(n)} \triangleq P_r\{(M_0, M_1) \neq (\hat{M}_0, \hat{M}_1)\}$

" " second " $P_{e_2}^{(n)} \triangleq P_r\{(M_0, M_2) \neq (\hat{M}_0, \hat{M}_2)\}$.

A rate triple (R_0, R_1, R_2) is said to be achievable if there exists a sequence of $(n, 2^{nR_0}, 2^{nR_1}, 2^{nR_2})$ codes such that the average error prob $P_e^{(n)} \xrightarrow[n \to \infty]{} 0$, where

$$P_e^{(n)} \triangleq \Pr\left\{ \left((M_0, M_1) \neq (\hat{M}_0, \hat{M}_1) \right) \cup \left((M_0, M_2) \neq (\hat{M}_0, \hat{M}_2) \right) \right\}.$$

a) Show that (R_0, R_1, R_2) is achievable if

$$R_1 \leq I(X; Y_1 | U)$$

$$R_0 + R_2 \leq I(U; Y_2)$$

$$R_0 + R_1 + R_2 \leq I(X; Y_1) \qquad — (1)$$

for some $p_{U,X}(\cdot, \cdot)$.

propose an encoder and decoders, and show that $P_e^{(n)} \to 0$ (as $n \to \infty$) if the above inequalities are satisfied. Hint: Use the idea of random coding and the idea of superposition codes.

b) Show that if (R_1, R_2) satisfies the private message constraint

$$R_1 \leq I(X; Y_1 | U)$$

$$R_2 \leq I(U; Y_2)$$

$$R_1 + R_2 \leq I(X; Y_1)$$

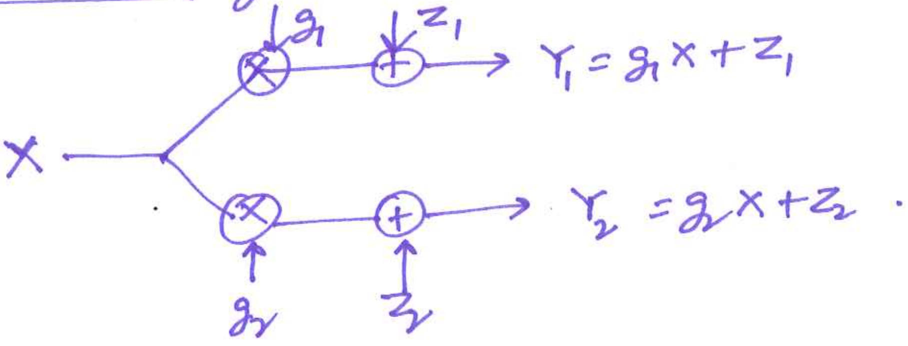
then $(R_0, R_1, R_2 - R_0)$ belongs to the common message set rate region given by (1) above. Assume

$$R_0 \leq \min(R_1, R_2).$$

Q.2. GMAC-GBC DUALITY

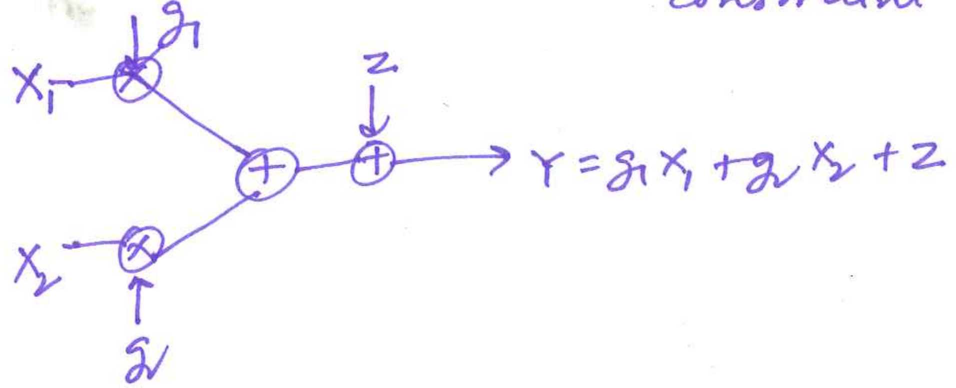
(25 points)

2-user ~~d~~ Gaussian BC



$E(|X|^2) \le P$, Z_1, Z_2 are i.i.d $\mathcal{CN}(0,1)$.

2-user Gaussian MAC with sum-power constraint.



$$E(|X_1|^2) + E(|X_2|^2) \le P.$$

$$Z \sim \mathcal{CN}(0,1)$$

a) Show that the capacity region of the 2-user GBC and the 2-user GMAC with sum-power constraint is same. Assume $|g_1| > |g_2|$.

b) Argue that the seq. of codes that achieves (R_1, R_2) on the boundary of the GMAC capacity region can be used to achieve the same point on the GBC.

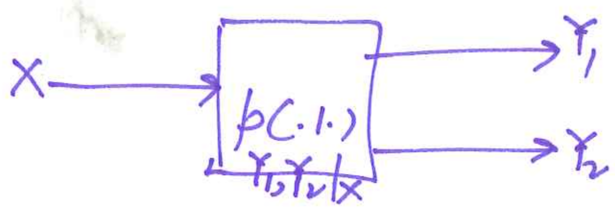
(4)

Q.3. SATO'S OUTER BOUND ON THE SUM RATE OF GENERAL BC's.

For the 2-user general BC (not necessarily degraded), show that

(10 points)

$$R_1 + R_2 \leq \min_{\mathcal{P}(p_{Y_1|X}(\cdot|\cdot),\, p_{Y_2|X}(\cdot|\cdot))} \max_{p_X(\cdot)} I(X; Y_1, Y_2).$$



Here, $\mathcal{P}(p_{Y_1|X}(), p_{Y_2|X}())$ = { set of all channel distribution $p'_{Y_1,Y_2|X}(\cdot|\cdot)$ such that $p'_{Y_1|X}(\cdot|\cdot) = p_{Y_1|X}(\cdot|\cdot)$ and $p'_{Y_2|X}(\cdot|\cdot) = p_{Y_2|X}(\cdot|\cdot)$ } i.e., having the same marginals as the original channel.

Q.4. Consider a 2-user Gaussian BC where the transmitter has N_t antennas and the users have single antenna each, i.e., (assume real valued channels)

(20 points)

$$Y_1 = \underline{g}_1^T \underline{X} + Z_1, \quad \underline{g}_1 = [g_{11}, \dots, g_{1N_t}]^T$$

$$Y_2 = \underline{g}_2^T \underline{X} + Z_2, \quad \underline{g}_2 = [g_{21}, \dots, g_{2N_t}]^T$$

Z_1, Z_2 are i.i.d $\mathcal{N}(0, N_0)$.

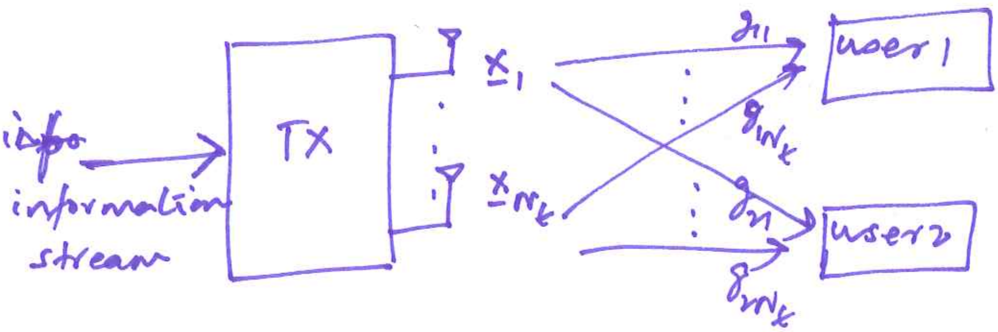
$\underline{X} \in \mathbb{R}^{N_t \times 1}$

Avg. Tx. power constraint:

$$\mathbb{E}\left(Tr\left(\underline{X}\,\underline{X}^T\right)\right) \leq P$$

$$G \triangleq \begin{bmatrix} \underline{g}_1 \\ \underline{g}_2 \end{bmatrix}$$

$$G \in \mathbb{R}^{2 \times N_t}$$



a) Is the above BC degraded? Justify your answer.

b) What is the capacity region if $\underline{g}_1 = \underline{g}_2$?

c) What is the maximum possible symmetric rate $R_1 = R_2 = R_{sym}$ if

$$G G^H = \begin{bmatrix} \|g_1\|^2 & 0 \\ 0 & \|g_2\|^2 \end{bmatrix}, \text{ and } \|g_1\|^2 = \|g_2\|^2.$$

Compare this with the maximum possible symmetric rate if $\underline{g}_1 = \underline{g}_2$ and $\|\underline{g}_1\| = \|\underline{g}_2\| = \|g\|$.

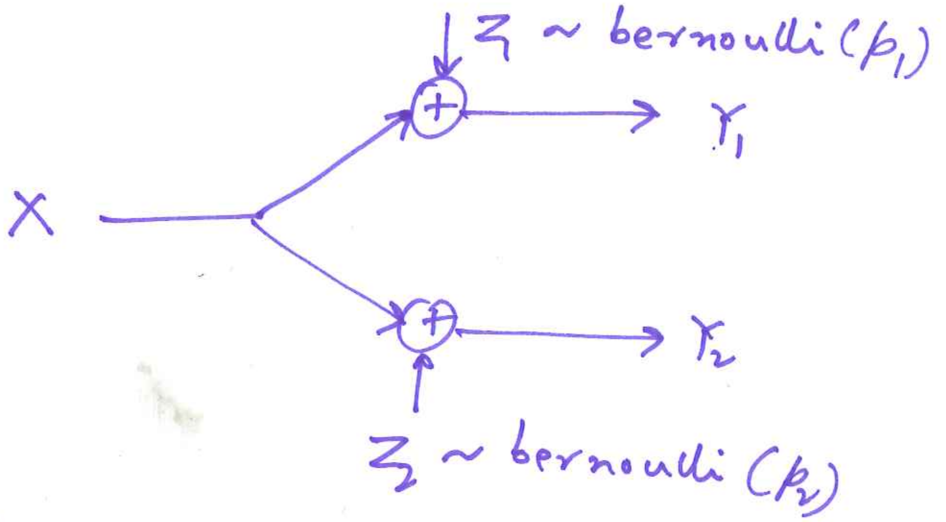
~~Is it better when~~ $\cancel{\underline{g}_1}$ ~~and~~ $\cancel{\underline{g}_2}$ ~~are~~
Which situation is better?

d) For $G G^H = \begin{bmatrix} \|g_1\|^2 & 0 \\ 0 & \|g_2\|^2 \end{bmatrix}$, ~~and~~ with $\|g_1\|^2 = \|g_2\|^2$,

what is the optimal TX and RX scheme. (Optimality is w.r.t. achieving the largest symmetric rate).

Q.5. Find the capacity region of the following Binary symmetric broadcast channel as shown below.

(20 points)



The channel input X is binary $(0, 1)$. Z_1 and Z_2 are independent bernoulli sequences. Z = bernoulli (p) means a ~~bin~~ distribution on the set $\{0, 1\}$ with prob $(Z=0) = 1-p$ and prob$(Z=1) = p$.

$\oplus$ signifies ~~the~~ modulo-2 addition.

Assume $p_1 < p_2 < \frac{1}{2}$.