Information Theory for Wireless Communications

Lecture 8: Channel Capacity of Waveform Channels

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In this lecture, we are interested in finding the capacity of the system depicted in Figure 1. We assume that all involved signals are real-valued and that $H(f)$ and $G(f)$ are linear time-invariant (LTI) filters with frequency responses given by $H(f)$ and $G(f)$ respectively. In the given system, the input signal $x(t)$ is assumed to be approximately time-limited to the interval $[-\frac{T}{2}, \frac{T}{2}]$ and approximately band-limited to the frequency band $[-B, B]$ with average power P. We further assume that the noise signal $w(t)$ is stationary white Gaussian random process, with mean zero and autocorrelation function $r_w(\tau) = \frac{N_0}{2}\delta(\tau)$.

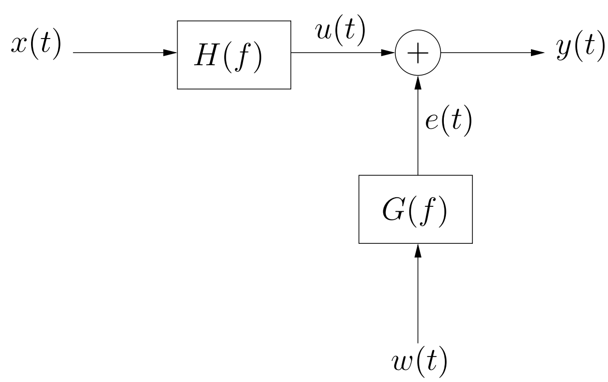

Fig. 1. Illustration of the system model considered in this lecture.

Since $x(t)$ is time-limited, we can use the Fourier series expansion to write $x(t)$ as

$$x(t) = \sum_{n=0}^{\infty} \bar{x}_n \bar{\phi}_n(t) + \sum_{n=1}^{\infty} \tilde{x}_n \tilde{\phi}_n(t), \tag{1}$$

where

$$\bar{\phi}_0(t) = \frac{1}{\sqrt{T}} \operatorname{rect}\left(\frac{t}{T}\right) = \begin{cases} \frac{1}{\sqrt{T}}, & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise,} \end{cases}$$

and for $n \geq 1$,

$$\bar{\phi}_n(t) = \begin{cases} \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi}{T} nt\right), & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{\phi}_n(t) = \begin{cases} \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi}{T} nt\right), & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\left\{\bar{\phi}_0(t), \bar{\phi}_n(t), \tilde{\phi}_n(t)\right\}_{n=1}^{\infty}$ form an orthonormal basis, since it can be easily shown that

$$\int_{-\infty}^{\infty} \bar{\phi}_n(t)\bar{\phi}_k(t)\mathrm{d}t = \delta_{n,k}, \quad \int_{-\infty}^{\infty} \tilde{\phi}_n(t)\tilde{\phi}_k(t)\mathrm{d}t = \delta_{n,k}, \quad \int_{-\infty}^{\infty} \bar{\phi}_n(t)\tilde{\phi}_n(t)\mathrm{d}t = 0.$$

The signal power can be also written as

$$P = \mathbb{E}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\mathrm{d}t\right] = \frac{1}{T}\,\mathbb{E}\left[\sum_{n=0}^{\infty} \bar{x}_n^2 + \sum_{n=1}^{\infty} \tilde{x}_n^2\right] = \frac{1}{T}\left(\sum_{n=0}^{\infty} \bar{p}_n + \sum_{n=1}^{\infty} \tilde{p}_n\right) \tag{2}$$

where $\bar{p}_n \triangleq \mathbb{E}\left[\bar{x}_n^2\right]$ and $\tilde{p}_n \triangleq \mathbb{E}\left[\tilde{x}_n^2\right]$.

Now consider the Fourier transform for $\bar{\phi}_0(t)$:

$$\begin{aligned} \bar{\Phi}_0(f) &= \int_{-\infty}^{\infty} \bar{\phi}_0(t)e^{-j2\pi ft}\mathrm{d}t = \sqrt{\frac{1}{T}}\int_{-T/2}^{T/2} e^{-j2\pi ft}\mathrm{d}t \\ &= \sqrt{T}\cdot\frac{\sin(\pi fT)}{\pi fT} = \sqrt{T}\cdot \mathrm{sinc}(\pi fT). \end{aligned} \tag{3}$$

Using this result and noting that

$$\bar{\phi}_n(t) = \sqrt{2}\cos\left(\frac{2\pi}{T}nt\right)\bar{\phi}_0(t), \qquad \tilde{\phi}_n(t) = \sqrt{2}\sin\left(\frac{2\pi}{T}nt\right)\bar{\phi}_0(t),$$

we can write (by using the properties of Fourier transform) that

$$\bar{\Phi}_n(f) = \sqrt{\frac{1}{2}}\left[\bar{\Phi}_0\left(f - \frac{n}{T}\right) + \bar{\Phi}_0\left(f + \frac{n}{T}\right)\right] = \sqrt{\frac{T}{2}}\Big[\mathrm{sinc}\big(\pi(fT-n)\big) + \mathrm{sinc}\big(\pi(fT+n)\big)\Big], \tag{4}$$

$$\tilde{\Phi}_n(f) = \frac{1}{j\sqrt{2}}\left[\bar{\Phi}_0\left(f - \frac{n}{T}\right) - \bar{\Phi}_0\left(f + \frac{n}{T}\right)\right] = \frac{1}{j}\cdot\sqrt{\frac{T}{2}}\Big[\mathrm{sinc}\big(\pi(fT-n)\big) - \mathrm{sinc}\big(\pi(fT+n)\big)\Big]. \tag{5}$$

The amplitude frequency responses $\bar{\Phi}_n(f)$ and $\tilde{\Phi}_n(f)$ are plotted in Figure 2. As we can see, $\bar{\phi}_n(t)$ and $\tilde{\phi}_n(t)$ have most of their energy in the frequency interval $[\frac{n-1}{T}, \frac{n+1}{T}]$. Therefore, we conclude that only $\bar{\phi}_n(t)$ and $\tilde{\phi}_n(t)$ with $n = (0), 1, 2, \ldots, BT$ have any significant energy in $[-B, B]$ and hence $x(t)$ can be well approximated by:

$$x(t) \approx \sum_{n=0}^{BT} \bar{x}_n\bar{\phi}_n(t) + \sum_{n=1}^{BT} \tilde{x}_n\tilde{\phi}_n(t). \tag{6}$$

In other words, $x(t)$ can be represented by roughly $2BT + 1 \approx 2BT$ real numbers. This number is often called the “degrees of freedom” (DoF) of signal $x(t)$. Thus we have the following important observation:

A signal with bandwidth B and time duration T has roughly $2BT$ degrees of freedom.

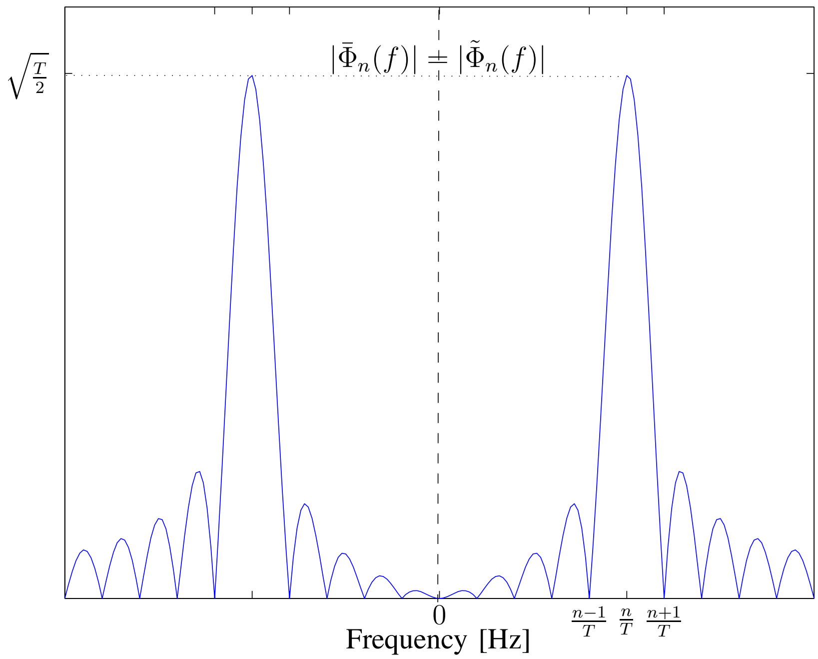

Fig. 2. Amplitude Fourier transform for $\bar{\phi}_n(t)$ and $\tilde{\phi}_n(t)$.

Now assume that $x(t)$ has no DC components, i.e.

$$\int_{-T/2}^{T/2} x(t)\mathrm{d}t = 0,$$

and thus $\bar{x}_0 = 0$, and assume that $T \gg$ time constants of the filters $H(f)$ and $G(f)$. In this case in order to find signals $u(t)$ and $e(t)$, we can use the fundamental result for LTI filters which states that the output of the LTI system corresponding to a sinusoidal input with frequency f_0 is a sinusoidal signal with the same frequency f_0 and with amplitude scaled by $|H(f_0)|$ and a phase shift of $\angle H(f_0)$, where $H(f)$ represents the frequency response of the LTI system.

$$\begin{aligned} u(t) &= \sum_{n=1}^{BT} \left[\bar{x}_n \left| H\left(n/T\right) \right| \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi}{T} nt + \angle H(n/T) \right) + \tilde{x}_n \left| H\left(n/T\right) \right| \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi}{T} nt + \angle H(n/T) \right) \right] \\ &= \sum_{n=1}^{BT} \left[\bar{u}_n \bar{\phi}_n(t) + \tilde{u}_n \tilde{\phi}_n(t) \right], \end{aligned} \tag{7}$$

where $\bar{u}_n \triangleq \bar{x}_n\bar{h}_n + \tilde{x}_n\tilde{h}_n$ and $\tilde{u}_n \triangleq -\bar{x}_n\tilde{h}_n + \tilde{x}_n\bar{h}_n$, with

$$\bar{h}_n \triangleq \Re\left\{H(n/T)\right\} = |H(n/T)|\cos\big(\angle H(n/T)\big),$$
$$\tilde{h}_n \triangleq \Im\left\{H(n/T)\right\} = |H(n/T)|\sin\big(\angle H(n/T)\big).$$

Similarly, we can use the Fourier expansion for the noise signal $w(t)$. We have

$$w(t) = \sum_{n=0}^{\infty} \bar{w}_n\bar{\phi}_n(t) + \sum_{n=1}^{\infty} \tilde{w}_n\tilde{\phi}_n(t). \tag{8}$$

Since the noise is assumed to be white Gaussian random process, $\bar{w}_n$ and $\tilde{w}_n$ are i.i.d.$\sim \mathcal{N}(0, N_0/2)$. Therefore by ignoring $\{\bar{w}_n, \tilde{w}_n\}_{n=BT+1}^{\infty}$ that has little energy in the frequency interval $[-B, B]$[1], we can write

$$e(t) = \sum_{n=1}^{BT}\left[\bar{e}_n\bar{\phi}_n(t) + \tilde{e}_n\tilde{\phi}_n(t)\right], \tag{9}$$

with $\bar{e}_n \triangleq \bar{w}_n\bar{g}_n + \tilde{w}_n\tilde{g}_n$ and $\tilde{e}_n \triangleq -\bar{w}_n\tilde{g}_n + \tilde{w}_n\bar{g}_n$ and

$$\bar{g}_n \triangleq \Re\left\{G(n/T)\right\} = |G(n/T)|\cos\big(\angle G(n/T)\big),$$
$$\tilde{g}_n \triangleq \Im\left\{G(n/T)\right\} = |G(n/T)|\sin\big(\angle G(n/T)\big).$$

We can also rewrite the above equations as

$$\begin{bmatrix}\bar{u}_n\\ \tilde{u}_n\end{bmatrix} = \underbrace{\begin{bmatrix}\bar{h}_n & \tilde{h}_n\\ -\tilde{h}_n & \bar{h}_n\end{bmatrix}}_{\triangleq H_n}\underbrace{\begin{bmatrix}\bar{x}_n\\ \tilde{x}_n\end{bmatrix}}_{\boldsymbol{x}_n}, \quad \text{and} \quad \begin{bmatrix}\bar{e}_n\\ \tilde{e}_n\end{bmatrix} = \underbrace{\begin{bmatrix}\bar{g}_n & \tilde{g}_n\\ -\tilde{g}_n & \bar{g}_n\end{bmatrix}}_{\triangleq G_n}\underbrace{\begin{bmatrix}\bar{w}_n\\ \tilde{w}_n\end{bmatrix}}_{\boldsymbol{w}_n}. \tag{10}$$

Now we can write $y(t)$ as

$$y(t) = u(t) + e(t) = \sum_{n=1}^{BT}\left[\underbrace{(\bar{u}_n + \bar{e}_n)}_{\triangleq\bar{y}_n}\bar{\phi}_n(t) + \underbrace{(\tilde{u}_n + \tilde{e}_n)}_{\triangleq\tilde{y}_n}\tilde{\phi}_n(t)\right]. \tag{11}$$

We next define

$$\boldsymbol{z}_n = \begin{bmatrix}\bar{z}_n\\ \tilde{z}_n\end{bmatrix} \triangleq H_n^{-1}\begin{bmatrix}\bar{y}_n\\ \tilde{y}_n\end{bmatrix}. \tag{12}$$

Note that this transformation is information lossless since we can always recover $\bar{y}_n$ and $\tilde{y}_n$ back from

[1]Note that we can remove those noise components by filtering them out at the receiver.

$\boldsymbol{z}_n$. We have that

$$\boldsymbol{z}_n = H_n^{-1}\left(H_n\boldsymbol{x}_n + G_n\boldsymbol{w}_n\right) = \boldsymbol{x}_n + \underbrace{\frac{1}{\bar{h}_n^2+\tilde{h}_n^2}H_n^T G_n\boldsymbol{w}_n}_{\triangleq \boldsymbol{q}_n}. \tag{13}$$

We see that $\boldsymbol{q}_n = [\bar{q}_n \ \tilde{q}_n]^T$ is Gaussian distributed with mean zero and covariance matrix

$$\mathbb{E}\left[\boldsymbol{q}_n\boldsymbol{q}_n^T\right] = \frac{1}{(\bar{h}_n^2+\tilde{h}_n^2)^2}H_n^T G_n\,\mathbb{E}\left[\boldsymbol{w}_n\boldsymbol{w}_n^T\right]G_n^T H_n = \frac{N_0}{2}\cdot\underbrace{\frac{\bar{g}_n^2+\tilde{g}_n^2}{\bar{h}_n^2+\tilde{h}_n^2}}_{\triangleq\lambda_n}\cdot\mathrm{I} \tag{14}$$

Therefore, $\bar{q}_n$ and $\tilde{q}_n$ are i.i.d.$\sim\mathcal{N}(0,\lambda_n N_0/2)$.

Now if we look at the above derivation in more details, we will realize that by using the given transformation, we get $2BT$ parallel discrete "tone" additive Gaussian sub-channels (AWG) with independent noises across different sub-channels of the form

$$\begin{cases}\bar{z}_n = \bar{x}_n + \bar{q}_n, \\ \tilde{z}_n = \tilde{x}_n + \tilde{q}_n,\end{cases} \qquad \bar{q}_n \text{ and } \tilde{q}_n \sim \mathcal{N}(0,\lambda_n N_0/2), \qquad \text{for} \quad n = 1,\ldots,2BT.$$

The capacities of the above sub-channels are $\frac{1}{2}\log_2\left(1+\frac{\bar{p}_n}{\lambda_n N_0/2}\right)$ and $\frac{1}{2}\log_2\left(1+\frac{\tilde{p}_n}{\lambda_n N_0/2}\right)$ respectively, and are achieved if $\bar{x}_n$ and $\tilde{x}_n$ are Gaussian distributed with mean zero and variance $\bar{p}_n$ and $\tilde{p}_n$, respectively. Thus we can finally find the capacity of the channel depicted in Figure 1 by solving the below optimization problem

$$C = \max\sum_{n=1}^{BT}\left\{\frac{1}{2}\log_2\left(1+\frac{\bar{p}_n}{\lambda_n N_0/2}\right)+\frac{1}{2}\log_2\left(1+\frac{\tilde{p}_n}{\lambda_n N_0/2}\right)\right\}, \tag{15}$$

subject to

$$\bar{p}_n \geq 0, \tag{16}$$

$$\tilde{p}_n \geq 0, \tag{17}$$

$$\sum_{n=1}^{BT}\left(\bar{p}_n+\tilde{p}_n\right) \leq PT. \tag{18}$$

Since the above problem is a convex optimization problem (meaning that the optimal solution is unique) and since it is symmetric with respect to $\bar{p}_n$ and $\tilde{p}_n$, we conclude that at the optimal solution $\bar{p}_n = \tilde{p}_n$

and thus the capacity can be found by solving

$$C = \max \sum_{n=1}^{BT} \left\{ \log_2 \left(1 + \frac{p_n}{\lambda_n N_0/2} \right) \right\}, \tag{19}$$

subject to

$$p_n \geq 0, \tag{20}$$

$$\sum_{n=1}^{BT} p_n \leq PT/2 \tag{21}$$

instead. Before we solve this problem in a general form, let us give the solution for the case where all λ_n:s are equal.

A. *Special Case of* $\lambda_n = 1$ *for all* $n = 1, \ldots, BT$:

In this case, because of the symmetry, the optimum solution is equal power allocation, i.e. $p_n = \frac{P}{2B}$ for all $n = 1, \ldots, BT$. In this case, the capacity becomes $C = BT \log_2 \left(1 + \frac{P}{BN_0}\right)$ bits. Since these many bits are transmitted during T seconds, the transmission rate in this case is $R = \frac{C}{T} = B \log_2 \left(1 + \frac{P}{BN_0}\right)$ bits/s. The spectral efficiency, which is defined as the number of bits transmitted during 1 second in 1 Hz of bandwidth is also $\eta = \frac{C}{BT} = \log_2 \left(1 + \frac{P}{BN_0}\right)$ bits/s/Hz for this case. Figure 3 illustrates the spectral efficiency as a function of signal-to-noise-ratio (SNR) $\gamma \triangleq \frac{P}{N_0 B}$ for a fixed bandwidth of B Hz.

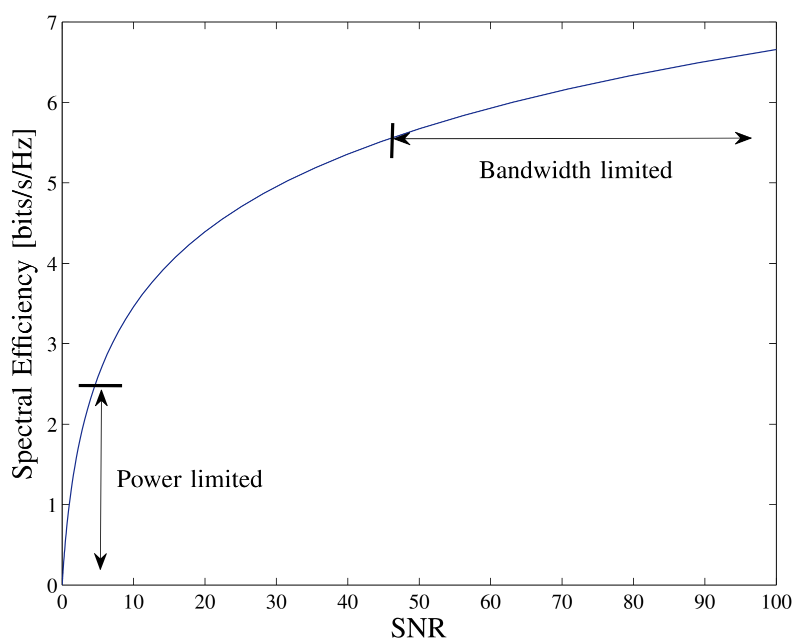

Fig. 3. Spectral efficiency as a function of SNR for a fixed bandwidth.

Note that for $x \ll 1$, we have $\log(1+x) \approx x$. Thus if SNR is very low, we can write $\eta \approx \gamma \log_2 e$. In other words, at low SNR region, the spectral efficiency scales linearly with SNR. This region is called the *power limited* region. On the other hand, if SNR is high, then $\eta = \log_2(1+\gamma) \approx \log_2(\gamma)$. That is every doubling of SNR results in 1 bit/s/Hz increase in spectral efficiency. This region is therefore called *bandwidth limited* region.

Now assume that the power P is fixed and we increase the bandwidth. We would like to find out the maximum achievable transmission rate R_∞. We have

$$R_\infty = \lim_{B\to\infty} R = \lim_{B\to\infty} B \log_2\left(1 + \frac{P}{N_0 B}\right) = \frac{P}{N_0}\log_2 e. \tag{22}$$

Figure 4 illustrates the transmission rate as a function of the bandwidth for the case with $P/N_0 = 1$.

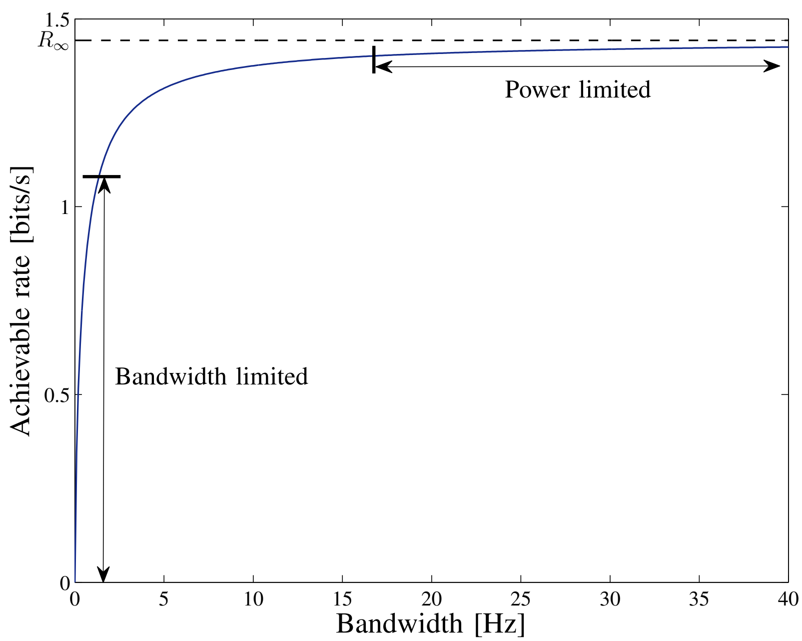

Fig. 4. Achievable rate as a function of bandwidth for a fixed P/N_0.

The energy spent for the transmission of one information bit E_b, is $E_b = P/R = P/(\eta B)$ joules, since with spending P watts power in T seconds we have sent $C = RT$ bits. Now using the fact that

$$\eta \le \log_2\left(1 + \frac{P}{N_0 B}\right) = \log_2\left(1 + \eta\frac{E_b}{N_0}\right),$$

we conclude that

$$\frac{E_b}{N_0} \ge \frac{2^\eta - 1}{\eta}. \tag{23}$$

Thus, we need at least $\frac{2^{\eta}-1}{\eta}$ SNR per information bit to communicate reliably. More particularly, by letting $\eta \to 0$, we can compute the minimum required SNR per bit for reliable communications to be

$$\left(\frac{E_b}{N_0}\right)_{\min} = \frac{1}{\log_2 e} \approx -1.59 \ \text{ dB}.$$

That is, one needs at least -1.59 dB energy per information bit to communicate reliably, which is a fundamental information theory limit.

B. Solution to the general case: waterfilling algorithm

In this section, we give the general solution to the optimization problems of the form (19). More precisely, we are interested in finding the optimal solution to the following problem:

$$\max \sum_{n=1}^{N} \log_2 \left(1 + \gamma_n \rho_n\right), \tag{24}$$

subject to:

$$\rho_n \geq 0, \forall n = 1, \ldots, N \tag{25}$$

$$\sum_{n=1}^{N} \rho_n \leq \rho. \tag{26}$$

We assume that γ_n:s are given and that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_N > 0$. First note that

$$\max \sum_{n=1}^{N} \log_2 \left(1 + \gamma_n \rho_n\right) \equiv \max \prod_{n=1}^{N} \left(1 + \gamma_n \rho_n\right) \equiv \max \left(\prod_{n=1}^{N} \gamma_n\right) \left(\prod_{n=1}^{N} \left(\frac{1}{\gamma_n} + \rho_n\right)\right). \tag{27}$$

Therefore, if any of the ρ_n:s are zero, these must be for ρ_n, $n = M+1, \cdots, N$, for some integer M, since otherwise one can still maximize the objective. Thus we can further simplify (27) using,

$$\begin{aligned} \left(\prod_{n=1}^{N} \gamma_n\right) \left(\prod_{n=1}^{N} \left(\frac{1}{\gamma_n} + \rho_n\right)\right) &= \left(\prod_{n=1}^{N} \gamma_n\right) \left(\prod_{n=1}^{M} \left(\frac{1}{\gamma_n} + \rho_n\right)\right) \left(\prod_{n=M+1}^{N} \frac{1}{\gamma_n}\right) \\ &= \left(\prod_{n=1}^{M} \gamma_n\right) \left(\prod_{n=1}^{M} \left(\frac{1}{\gamma_n} + \rho_n\right)\right) \end{aligned} \tag{28}$$

Using the inequality of arithmetic and geometric means:

$$\frac{1}{N} \sum_{n=1}^{N} x_n \geq \left(\prod_{n=1}^{N} x_n\right)^{\frac{1}{N}}, \qquad x_n \geq 0. \tag{29}$$

with equality if and only if are x_n:s are equal, we can write

$$\prod_{n=1}^{M}\left(\frac{1}{\gamma_n}+\rho_n\right)\leq\left(\frac{1}{M}\sum_{n=1}^{M}\left(\frac{1}{\gamma_n}+\rho_n\right)\right)^M\leq\left(\frac{1}{M}\sum_{n=1}^{M}\frac{1}{\gamma_n}+\frac{1}{M}\rho\right)^M, \tag{30}$$

with equality if

$$\begin{cases}\frac{1}{\gamma_n}+\rho_n=\text{const.}=\mu\\ \sum_{n=1}^{N}\rho_n=\rho\end{cases}$$

where the first and the second constraints are due to the first and the second inequalities in (30), respectively. Therefore, the solution to the optimization problem is

$$\begin{cases}\rho_n^*=\max\left\{0,\mu-\frac{1}{\gamma_n}\right\},\\ \mu=\frac{1}{M}\left(\rho+\sum_{n=1}^{M}\frac{1}{\gamma_n}\right)\end{cases} \tag{31}$$

Note that to find M, that is to find the number of non-zero ρ_n:s, we can solve the above equations iteratively. More precisely, we first fix μ, then we compute ρ_n:s from (31), and if $\sum_{n=1}^{N}\rho_n=\rho$, then we return ρ_n:s as the final solution, otherwise if $\sum_{n=1}^{N}\rho_n<(>)\rho$, we further increase (decrease) μ and repeat the procedure. This procedure is summarized in Algorithm 1.

Algorithm 1 Waterfilling algorithm.

1. Set μ to an arbitrary value;
2. Set $\rho_n=\max\left\{0,\mu-\frac{1}{\gamma_n}\right\}$, $\forall n=1,\ldots,N$;
3. Compute $t=\sum_{n=1}^{N}\rho_n$;

if $t<\rho$ **then**
  Set $\mu=\mu+\delta_\mu$;
  Goto 2;
else if $t>\rho$ **then**
  Set $\mu=\mu-\delta_\mu$;
  Goto 2;
else
  return ρ_n;
end if

This procedure is known as *waterfilling* algorithm and the μ at the optimum solution is called the *water-level*. The reason for this name can be understood from Figure 5, where we have plotted the optimal ρ_n^* for an illustrative example with $N=5$. As can be seen from this example, the waterfilling algorithm can be thought of as filling water into a tank containing N cylinders with heights equal to $\frac{1}{\gamma_n}$. Then μ will be the water-level and the height of water above each cylinder n will be the optimal solution ρ_n^*.

Let us finally apply the waterfilling algorithm to our problem. We have $\rho=\frac{PT}{2}$ and $\gamma_n=\frac{1}{\lambda_n N_0/2}$. The

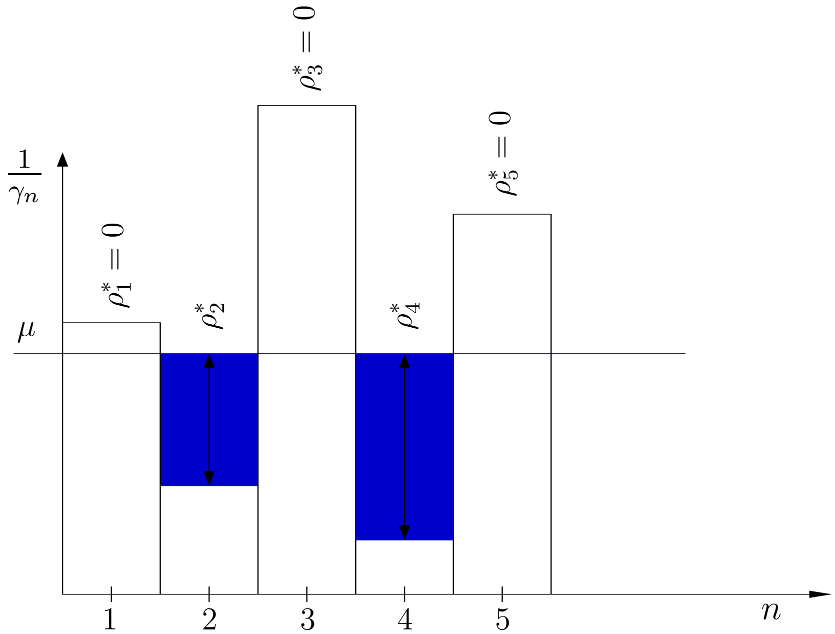

Fig. 5. Waterfilling algorithm for an illustrative example with $N = 5$.

interpretation of the waterfilling algorithm in this problem is that we assign more power to the "good" channels, i.e. the channels for whom the noise variance $\lambda_n N_0/2$ is smaller and "bad" channels may not even be used.

On a final note, let us consider two extreme cases: (i) P is very small (low SNR regime), and (ii) P is very large (high SNR regime). In case (i), as it can be seen from Figure 6 the optimum power allocation is to allocate all the power to the best sub-channel.

$$p_n = \begin{cases} \frac{PT}{2}, & n = \underset{k=1}{\overset{N}{\operatorname{argmin}}} \quad \lambda_k \\ 0, & \text{otherwise} \end{cases} \tag{32}$$

In this case, the capacity is $C = \log_2\left(1 + \frac{PT}{\lambda_{\min} N_0}\right) \approx \frac{PT}{\lambda_{\min} N_0} \log_2 e$. In high SNR regime, the optimum solution is roughly equal power allocation, as can be seen from Figure 7.

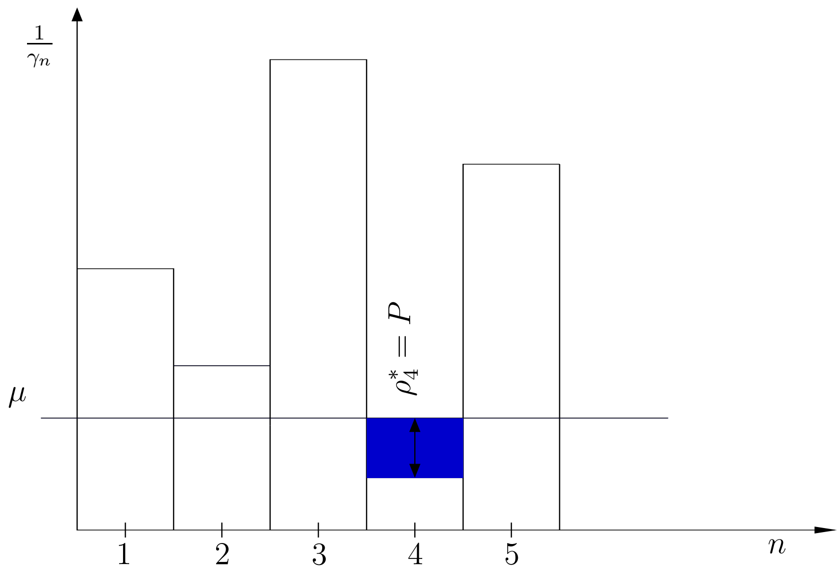

Fig. 6. Waterfilling algorithm for an illustrative example with $N = 5$ and for low SNR regime.

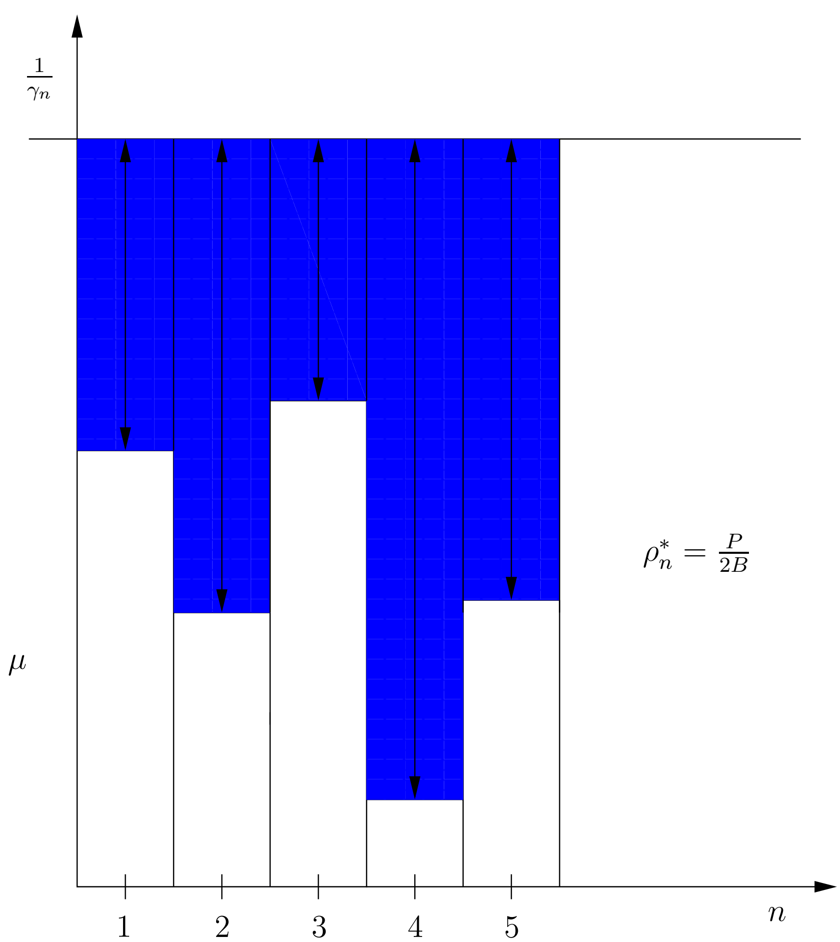

Fig. 7. Waterfilling algorithm for an illustrative example with $N = 5$ and for high SNR regime.