Information Theory for Wireless Communication

Lecture 12: Single-antenna Multi-user Uplink Channels

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In this lecture, we introduce the capacity region of the discrete memoryless multiple access channel (DM-MAC). We limit our lecture to the two-sender MAC as shown in Fig. 1. Two senders transmit their information simultaneously to a common receiver. Each Sender i, $i = 1, 2$, independently encodes its data m_i into a codeword $X_i(m_i)$ and then transmit to the receiver. The receiver received signal Y^n which is the combination of signals transmitted from both senders. Then, the receiver estimate the transmitted signals $\hat{m}_1$ and $\hat{m}_2$.

Definition 1 (DM-MAC)**.** A discrete memoryless 2-sender MAC is represented by

$$(\mathcal{X}_1, \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$$

where $\mathcal{X}_1$, $\mathcal{X}_2$ are input alphabets, $\mathcal{Y}$ is output alphabet, and $p(y|x_1, x_2)$ is the transition probability.

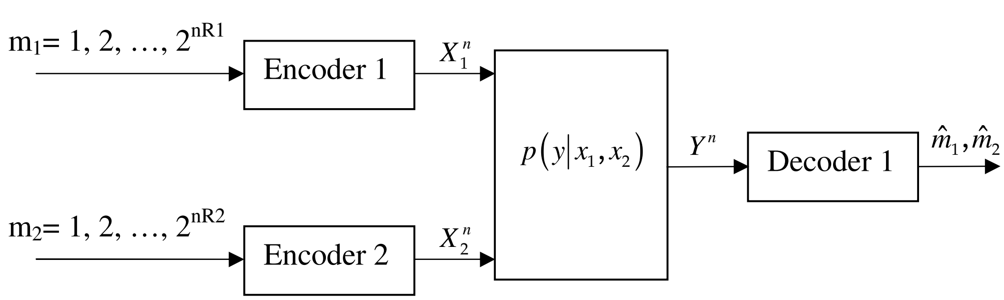

Figure 1. Two-sender Multiple access communication systems.

I. Simple Bounds on Capacity Region

We consider simple bound for individual rates and sum-rate.

- Bounds for individual rates:

  The achievable individual rates are upper bounded by

$$R_1 \leq C_1 = \max_{p_{X_1}(x_1), x_2 \in \mathcal{X}_2} I\left(X_1; Y | X_2 = x_2\right) \quad (1)$$

$$R_2 \leq C_1 = \max_{p_{X_2}(x_2), x_1 \in \mathcal{X}_1} I\left(X_2; Y | X_1 = x_1\right). \quad (2)$$

  The bound C_1 (or C_2) can be achieved by assuming that the receiver know X_2 (or X_1) prior to decoding.

- Bounds for the sum-rate:

  By using the fact that the sum-rate for MAC is always smaller than or equal to the sum-rate obtained when Sender 1 and Sender 2 cooperate in sending their data. More precisely,

$$R_1 + R_2 \leq C_{12} = \max_{p_{X_1}(x_1), p_{X_2}(x_2)} I\left(X_1, X_2; Y\right). \quad (3)$$

From (1), (2), and (3), we can obtain an outer bound on the capacity region of the MAC with 2 senders as

$$\begin{aligned} R_1 &\leq C_1 = \max_{p_{X_1}(x_1), x_2 \in \mathcal{X}_2} I\left(X_1; Y | X_2 = x_2\right) \\ R_2 &\leq C_1 = \max_{p_{X_2}(x_2), x_1 \in \mathcal{X}_1} I\left(X_2; Y | X_1 = x_1\right) \\ R_1 + R_2 &\leq C_{12} = \max_{p_{X_1}(x_1), p_{X_2}(x_2)} I\left(X_1, X_2; Y\right). \end{aligned}$$

Fig. 2 shows the outer bound as well as the inner bound of the capacity region. Here, the inner bound is obtained by using time division (or frequency division) schemes.

In next section, we will show that this outer bound coincides with the capacity region of the DM-MAC.

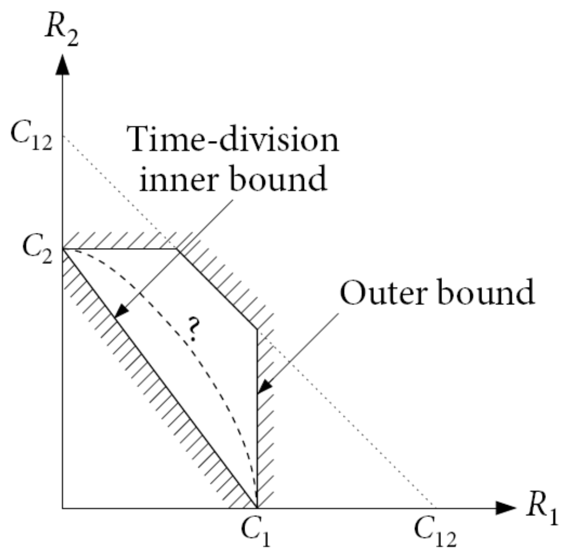

Figure 2. Inner bound and an outer bound on the capacity region of 2-sender DM-MAC.

II. Capacity Region of DM-MAC

The capacity region of the DM-MAC can be characterized as following theorem.

Theorem 1. *The capacity of a multiple-access channel* $(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$ *is the closure of the convex hull of all* $(R1, R2)$ *satisfying*

$$R_1 \leq C_1 = \max_{p_{X_1}(x_1), x_2 \in \mathcal{X}_2} I(X_1; Y | X_2 = x_2)$$

$$R_2 \leq C_1 = \max_{p_{X_2}(x_2), x_1 \in \mathcal{X}_1} I(X_2; Y | X_1 = x_1)$$

$$R_1 + R_2 \leq C_{12} = \max_{p_{X_1}(x_1), p_{X_2}(x_2)} I(X_1, X_2; Y)$$

for some product distribution $P(x_1)p(x_2)$ *on* $\mathcal{X}_1 \times \mathcal{X}_2$*.*

A. Proof of Achievability

- Codebook generation: create two random codebooks
  1. Fix input distribution $p(x_1, x_2) = p(x_1)p(x_2)$.
  2. Sender 1 generates 2^{nR_1} independent codewords $X_1^n(m_1)$, $m_1 \in \left(1, 2, ..., 2^{nR_1}\right)$.
  3. Sender 2 generates 2^{nR_2} independent codewords $X_2^n(m_2)$, $m_2 \in \left(1, 2, ..., 2^{nR_2}\right)$.

- Encoding:

  To send m_1 and m_2, Senders 1 and 2 send codewords $X_1^n(m_1)$ and $X_2^n(m_2)$, respectively.

- Decoding: denote by $\mathcal{T}_\epsilon^n$ the typical sets.

  The decoder uses successive cancellation techniques to detect the transmitted signals from two senders, i.e.,

  1. The receiver decodes m_1 first. If there exists a unique $\hat{m}_1$ such that $\left(X_1^n(\hat{m}_1), y^n\right) \in \mathcal{T}_\epsilon^n$, then the receiver performs step 2. Otherwise an error is declared.

  2. Cancellation: decode m_2 using $\hat{m}_1$.

  The receiver declares that a message m_2 was sent if there exists a unique $\hat{m}_2$ such that

$$\left(X_1^n(\hat{m}_1), X_2^n(\hat{m}_2), y^n\right) \in \mathcal{T}_\epsilon^n.$$

- Average probability of error:

  By symmetry, we assume $(m_1 = 1, m_2 = 1)$ is sent. Then the average probability of error is given by

$$P_e^n = \mathtt{Pr}\left(\text{error}|m_1 = 1, m_2 = 1\right) \tag{4}$$

  Define the event

$$\mathcal{E}_{ij} \triangleq \left\{\left(X_1^n(i), X_2^n(j), y^n\right) \in \mathcal{T}_\epsilon^n\right\}.$$

  Then, we have

$$P_e^{(n)} = \mathtt{Pr}\left(\mathcal{E}_{11}^c \bigcup_{(i,j)\neq(1,1)} \mathcal{E}_{ij}\right) \le \mathtt{Pr}\left(\mathcal{E}_{11}^c\right) + \sum_{i\neq 1} \mathtt{Pr}\left(\mathcal{E}_{i1}\right) + \sum_{j\neq 1} \mathtt{Pr}\left(\mathcal{E}_{1j}\right) + \sum_{i\neq 1, j\neq 1} \mathtt{Pr}\left(\mathcal{E}_{ij}\right). \tag{5}$$

  Clearly, from the AEP, $\mathtt{Pr}\left(\mathcal{E}_{11}^c \bigcup_{(i,j)\neq(1,1)} \mathcal{E}_{ij}\right) \to 0$, as $n \to \infty$. We now consider remaining terms of (5).

+ For $i \neq 1,\ j = 1$, we have

$$\begin{aligned}
\mathtt{Pr}\left(\mathcal{E}_{i1}\right) &= \mathtt{Pr}\left(\left(X_1^n\left(i\right), X_2^n\left(1\right), y^n\right) \in \mathcal{T}_\epsilon^n\right) && (6)\\
&= \sum_{\left(x_1^n(i), x_2^n(1), y^n\right) \in \mathcal{T}_\epsilon^n} p\left(x_1^n\right) p\left(x_2^n, y^n\right) && (7)\\
&\leq |\mathcal{T}_\epsilon^n| 2^{-n(H(X_1)-\epsilon)} 2^{-n(H(X_2,Y)-\epsilon)} && (8)\\
&\leq 2^{-n(H(X_1)+H(X_2,Y)-H(X_1,X_2,Y)-3\epsilon)} && (9)\\
&= 2^{-n(I(X_1;X_2,Y)-3\epsilon)} && (10)\\
&= 2^{-n(I(X_1;Y|X_2)-3\epsilon)} && (11)
\end{aligned}$$

where the last equality follows from the fact that X_1 and X_2 are independent, and the consequent $I(X_1;X_2,Y) = I(X_1;Y|X_2) + I(X_1;X_2) = I(X_1;Y|X_2)$.

+ For $i = 1,\ j \neq 1$, similarly, we have

$$\mathtt{Pr}\left(\mathcal{E}_{1j}\right) \leq 2^{-n(I(X_2;Y|X_1)-3\epsilon)} \tag{12}$$

+ For $i \neq 1,\ j \neq 1$, we have

$$\mathtt{Pr}\left(\mathcal{E}_{ij}\right) \leq 2^{-n(I(X_1,X_2;Y)-4\epsilon)}. \tag{13}$$

Substituting (11), (12), and (13) into (5), we have

$$P_e^{(n)} \leq \epsilon + 2^{nR_1} 2^{-n(I(X_1;Y|X_2)-3\epsilon)} + 2^{nR_2} 2^{-n(I(X_2;Y|X_1)-3\epsilon)} + 2^{n(R_1+R_2)} 2^{-n(I(X_1,X_2;Y)-4\epsilon)}. \tag{14}$$

We can see that with the conditions of Theorem 1, the average error probability tends to 0 when n goes to infinity. Therefore, we completes the proof of the achievability.