Note for Cramer-Rao Bounds

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This note provides some techniques for finding CRBs: Constrained CRBs [1], Projector methods for finding CRBs [2], and Concentrated CRBs [3].

I. Notation

- $\boldsymbol{X} \geq \boldsymbol{Y}$ means $\boldsymbol{X} - \boldsymbol{Y}$ is nonnegative definite
- $(z)_r$ and $(z)_i$ denote the real and imaginary part of z.

II. Constrained Cramer-Rao Bound

A. Problem Statement

Problem statement and notation are based on [1].

- $\boldsymbol{a}$: a $K \times 1$ non-random vector which are to be estimated.
- $\boldsymbol{r}$: an observation of a random vector .
- $\hat{\boldsymbol{a}}(\boldsymbol{R})$: an estimate of $\boldsymbol{a}$ basing on the observed vector $\boldsymbol{r}$. It is required that $\hat{\boldsymbol{a}}(\boldsymbol{R})$ satisfies M nonlinear equality constraints ($M < K$), as

$$\boldsymbol{f}\left(\hat{\boldsymbol{a}}(\boldsymbol{R})\right) = \mathbf{0}_{M\times 1} \tag{1}$$

$\Rightarrow$ **Problem is finding the CRB for the above estimator.**

B. Constrained Cramer-Rao Bound

Notation and Assumption:

- $\boldsymbol{F}(\boldsymbol{A})$: $K \times M$ gradient matrix of the constraints

$$\boldsymbol{F}(\boldsymbol{A}) \triangleq \nabla_{\boldsymbol{A}} \boldsymbol{f}^T(\boldsymbol{A}) \tag{2}$$

- $\boldsymbol{J}$: $K \times K$ Fisher information matrix

$$\boldsymbol{J} \triangleq \mathbb{E}\left\{\nabla_{\boldsymbol{A}} \ln p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A}) \nabla_{\boldsymbol{A}}^T \ln p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A})\right\} \tag{3}$$

- $\boldsymbol{B}(\boldsymbol{A})$: the bias of the estimate ($K \times 1$ vector)

$$\boldsymbol{B}(\boldsymbol{A}) \triangleq \mathbb{E}\left\{\hat{\boldsymbol{a}}(\boldsymbol{R})\right\} - \boldsymbol{A} = \int \left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right] p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A})\, d\boldsymbol{R} \tag{4}$$

$\Rightarrow$ The constrained CRB for unbiased estimator is given by

$$\mathbb{E}\left\{\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right]\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right]^T\right\} \geq \boldsymbol{J}^{-1} - \boldsymbol{J}^{-1}\boldsymbol{F}\left(\boldsymbol{F}^T\boldsymbol{J}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^T\boldsymbol{J}^{-1} \tag{5}$$

The result (5) shows that the constrained CRB is a nonsingular matrix that is smaller than the unconstrained CRB.

C. Summarize the Proof of Constrained CRB

1. Taking the gradient of both sides of (4), we obtain

$$\left[\nabla_{\boldsymbol{A}} \boldsymbol{B}^T(\boldsymbol{A})\right]^T = \mathbb{E}\left\{\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right] \nabla_{\boldsymbol{A}}^T \ln p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A})\right\} - \boldsymbol{I}_K \tag{6}$$

2. Showing that

$$\left[\nabla_{\boldsymbol{A}} \boldsymbol{B}^T(\boldsymbol{A})\right]^T \boldsymbol{\delta} = \boldsymbol{0}_{K\times 1} \tag{7}$$

   where $\boldsymbol{\delta}$ is a matrix satisfying $\boldsymbol{F}^T(\boldsymbol{A})\,\boldsymbol{\delta} = \boldsymbol{0}$.

   (Note that for unconstrained unbiased estimator, (7) holds for all $\boldsymbol{\delta}$)

3. Multiplying both sides of (6) by $\boldsymbol{\delta}$, and using (7), we have

$$\mathbb{E}\left\{\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right] \nabla_{\boldsymbol{A}}^T \ln p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A})\right\} \boldsymbol{\delta} = \boldsymbol{\delta} \tag{8}$$

4. By choosing $\boldsymbol{\delta} = \left[\boldsymbol{J}^{-1} - \boldsymbol{J}^{-1}\boldsymbol{F}\left(\boldsymbol{F}^T\boldsymbol{J}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^T\boldsymbol{J}^{-1}\right]\boldsymbol{\epsilon}$ for some arbitrary $K \times 1$ vector, we have

$$\begin{aligned}\mathbb{E}\left\{\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right] \nabla_{\boldsymbol{A}}^T \ln p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A})\right\} &\left[\boldsymbol{J}^{-1} - \boldsymbol{J}^{-1}\boldsymbol{F}\left(\boldsymbol{F}^T\boldsymbol{J}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^T\boldsymbol{J}^{-1}\right]\boldsymbol{\epsilon} \\ &= \left[\boldsymbol{J}^{-1} - \boldsymbol{J}^{-1}\boldsymbol{F}\left(\boldsymbol{F}^T\boldsymbol{J}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^T\boldsymbol{J}^{-1}\right]\boldsymbol{\epsilon}\end{aligned} \tag{9}$$

   Since (9) holds for all $\boldsymbol{\epsilon}$, we obtain

$$\mathbb{E}\left\{\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right] \nabla_{\boldsymbol{A}}^T \ln p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A})\right\} \boldsymbol{K} = \boldsymbol{K} \tag{10}$$

   where $\boldsymbol{K} \triangleq \left[\boldsymbol{J}^{-1} - \boldsymbol{J}^{-1}\boldsymbol{F}\left(\boldsymbol{F}^T\boldsymbol{J}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^T\boldsymbol{J}^{-1}\right]$.

5. Using the fact that any linear combination of the error vector and the gradient of the log-likelihood function is a zero-mean random variable whose covariance matrix must be nonnegative definite, we obtain

$$\begin{aligned}&\mathbb{E}\left\{\left(\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right] - \boldsymbol{K}\nabla_{\boldsymbol{A}} \ln p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A})\right)\left(\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right] - \boldsymbol{K}\nabla_{\boldsymbol{A}} \ln p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A})\right)^T\right\} \geq \boldsymbol{0} \\ &\Leftrightarrow \mathbb{E}\left\{\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right]\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right]^T\right\} - 2\,\mathbb{E}\left\{\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right] \nabla_{\boldsymbol{A}}^T \ln p_{\boldsymbol{r}|\boldsymbol{a}}(\boldsymbol{R}|\boldsymbol{A})\right\} \boldsymbol{K} \\ &\qquad + \boldsymbol{KJK} \geq \boldsymbol{0}\end{aligned} \tag{11}$$

   Using (10) and the fact that $\boldsymbol{KJK} = K$, from (11) we have

$$\mathbb{E}\left\{\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right]\left[\hat{\boldsymbol{a}}(\boldsymbol{R}) - \boldsymbol{A}\right]^T\right\} - \boldsymbol{K} \geq \boldsymbol{0} \tag{12}$$

   We then complete the proof.

III. Projector Methods for Fiding Neat Proofs of CRBs

This section provides proofs of CRBs for principal parameters (the parameters of interest).

A. Problem Statement

Consider the following superimposed-signals-in-noise model

$$\boldsymbol{y} = \boldsymbol{S}(\boldsymbol{\omega})\,\boldsymbol{a} + \boldsymbol{e} \tag{13}$$

$\boldsymbol{y} \in \mathbb{C}^{N\times 1}$: vector of observations

$\boldsymbol{e} \in \mathbb{C}^{N\times 1}$: noise

$\boldsymbol{a} \in \mathbb{C}^{n\times 1}$: vector of nuisance parameters, $\boldsymbol{a} = [a_1 \;...\; a_n]^T$

$\boldsymbol{\omega} \in \mathbb{R}^{n\times 1}$: vector of principal parameters, $\boldsymbol{\omega} \triangleq [\omega_1 \;...\; \omega_n]^T$

$\boldsymbol{S}(\boldsymbol{\omega}) \in \mathbb{C}^{N\times n}$ (assuming full rank) is denoted by

$$\boldsymbol{S}(\boldsymbol{\omega}) \triangleq [\boldsymbol{s}(\omega_1) \;\; ... \; \boldsymbol{s}(\omega_n)] \tag{14}$$

The unknown parameters are $\boldsymbol{a}$ and $\boldsymbol{\omega}$, but we are only interested in $\boldsymbol{\omega}$ and hence, we focus on the CRB for the principal parameter vector $\boldsymbol{\omega}$. The CRB for $\boldsymbol{\omega}$ is given by

$$\mathrm{CRB}_{\boldsymbol{\omega}} = \frac{\sigma^2}{2}\left[\left(\boldsymbol{D}^H \Pi_{\boldsymbol{S}}^{\perp} \boldsymbol{D}\right)_r\right]^{-1} \tag{15}$$

We will consider some methods to derive the above CRB.

B. Method 1

The main idea of this method come from the fact that the CRB for $\boldsymbol{\omega}$ is the $n \times n$ lower right block of the CRB for $\boldsymbol{\theta} \triangleq \left[\boldsymbol{a}_r^T \; \boldsymbol{a}_i^T \; \boldsymbol{\omega}^T\right]^T$ which is given by

$$\mathrm{CRB} = \frac{\sigma^2}{2}\left[\left(\boldsymbol{G}^H \boldsymbol{G}\right)_r\right]^{-1} \tag{16}$$

where

$$\boldsymbol{G} \triangleq \left[\frac{\partial \boldsymbol{S}(\boldsymbol{\omega})\,\boldsymbol{a}}{\boldsymbol{a}_r^T} \;\; \frac{\partial \boldsymbol{S}(\boldsymbol{\omega})\,\boldsymbol{a}}{\boldsymbol{a}_i^T} \;\; \frac{\partial \boldsymbol{S}(\boldsymbol{\omega})\,\boldsymbol{a}}{\boldsymbol{\omega}^T}\right] = [\boldsymbol{S} \;\; i\boldsymbol{S} \;\; \boldsymbol{D}] \tag{17}$$

where $\boldsymbol{D} \triangleq [\boldsymbol{s}'(\omega_1)\, a_1 \; ... \; \boldsymbol{s}'(\omega_n)\, a_n]$.

To find $n \times n$ lower right block of the CRB in (16), we use the following formulas

- Matrix inversion formula in block form

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^{-1} = \begin{bmatrix} \left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}\right)^{-1} & -\boldsymbol{A}^{-1}\boldsymbol{B}\left(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} \\ -\boldsymbol{D}^{-1}\boldsymbol{C}\left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}\right)^{-1} & \left(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} \end{bmatrix} \tag{18}$$

- Real and Imaginary part expressions

$$a_r b_r - a_i b_i = (ab)_r \tag{19}$$

$$a_i b_r + a_r b_i = (ab)_i \tag{20}$$

Detailed steps for the derivation of (15) are as follows

1. From (17), we obtain

$$\left(\boldsymbol{G}^H\boldsymbol{G}\right)_r = \begin{bmatrix} \left(\boldsymbol{S}^H\boldsymbol{S}\right)_r & -\left(\boldsymbol{S}^H\boldsymbol{S}\right)_i & \left(\boldsymbol{S}^H\boldsymbol{D}\right)_r \\ \left(\boldsymbol{S}^H\boldsymbol{S}\right)_i & \left(\boldsymbol{S}^H\boldsymbol{S}\right)_r & \left(\boldsymbol{S}^H\boldsymbol{D}\right)_i \\ \left(\boldsymbol{D}^H\boldsymbol{S}\right)_r & -\left(\boldsymbol{D}^H\boldsymbol{S}\right)_i & \left(\boldsymbol{D}^H\boldsymbol{D}\right)_r \end{bmatrix} \tag{21}$$

2. Using the the matrix inversion formula in block form (18), we have

$$\mathrm{CRB}_{\boldsymbol{\omega}} = \frac{\sigma^2}{2}\left\{\left(\boldsymbol{D}^H\boldsymbol{D}\right)_r - \left[\left(\boldsymbol{D}^H\boldsymbol{S}\right)_r \quad -\left(\boldsymbol{D}^H\boldsymbol{S}\right)_i\right] \begin{bmatrix} \left(\boldsymbol{S}^H\boldsymbol{S}\right)_r & -\left(\boldsymbol{S}^H\boldsymbol{S}\right)_i \\ \left(\boldsymbol{S}^H\boldsymbol{S}\right)_i & \left(\boldsymbol{S}^H\boldsymbol{S}\right)_r \end{bmatrix}^{-1} \begin{bmatrix} \left(\boldsymbol{S}^H\boldsymbol{D}\right)_r \\ \left(\boldsymbol{S}^H\boldsymbol{D}\right)_i \end{bmatrix}\right\}^{-1} \tag{22}$$

$$\stackrel{(a)}{=} \frac{\sigma^2}{2}\left\{\left(\boldsymbol{D}^H\boldsymbol{D}\right)_r - \left[\left(\boldsymbol{D}^H\boldsymbol{S}\right)_r \quad -\left(\boldsymbol{D}^H\boldsymbol{S}\right)_i\right] \begin{bmatrix} \left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\right]_r & -\left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\right]_i \\ \left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\right]_i & \left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\right]_r \end{bmatrix} \begin{bmatrix} \left(\boldsymbol{S}^H\boldsymbol{D}\right)_r \\ \left(\boldsymbol{S}^H\boldsymbol{D}\right)_i \end{bmatrix}\right\}^{-1} \tag{23}$$

$$\stackrel{(b)}{=} \frac{\sigma^2}{2}\left\{\left(\boldsymbol{D}^H\boldsymbol{D}\right)_r - \left[\left(\boldsymbol{D}^H\boldsymbol{S}\right)_r \quad -\left(\boldsymbol{D}^H\boldsymbol{S}\right)_i\right] \begin{bmatrix} \left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\boldsymbol{S}^H\boldsymbol{D}\right]_r \\ \left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\boldsymbol{S}^H\boldsymbol{D}\right]_i \end{bmatrix}\right\}^{-1} \tag{24}$$

$$= \frac{\sigma^2}{2}\left\{\left(\boldsymbol{D}^H\boldsymbol{D}\right)_r - \left[\boldsymbol{D}^H\Pi_{\boldsymbol{S}}\boldsymbol{D}\right]\right\}^{-1} \tag{25}$$

$$= \frac{\sigma^2}{2}\left[\left(\boldsymbol{D}^H\Pi_{\boldsymbol{S}}^{\perp}\boldsymbol{D}\right)_r\right]^{-1} \tag{26}$$

where (a) comes from the fact that

$$\begin{bmatrix} \left(\boldsymbol{S}^H\boldsymbol{S}\right)_r & -\left(\boldsymbol{S}^H\boldsymbol{S}\right)_i \\ \left(\boldsymbol{S}^H\boldsymbol{S}\right)_i & \left(\boldsymbol{S}^H\boldsymbol{S}\right)_r \end{bmatrix}^{-1} = \begin{bmatrix} \left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\right]_r & -\left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\right]_i \\ \left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\right]_i & \left[\left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\right]_r \end{bmatrix} \tag{27}$$

and (b) is obtained by using (19) and (20).

C. Method 2

The idea of this method is linearizing the model (13) as follows

$$\boldsymbol{z} = \boldsymbol{G}\boldsymbol{\theta} + \boldsymbol{\epsilon} \tag{28}$$

where $\boldsymbol{\theta} = \left[\boldsymbol{a}_r^T \ \boldsymbol{a}_i^T \ \boldsymbol{\omega}^T\right]^T$, and using the fact that the CRB for $\boldsymbol{\theta}$ is also the covariance matrix of the least-square estimate (LS) of $\boldsymbol{\theta}$.

Detailed steps:

1. From (17), we rewrite (28) as

$$\boldsymbol{z} = \boldsymbol{S}\boldsymbol{a}_r + i\boldsymbol{S}\boldsymbol{a}_i + \boldsymbol{D}\boldsymbol{\omega} + \boldsymbol{\epsilon} \tag{29}$$

2. Representing $\boldsymbol{D\omega}$ by $\Pi_{\boldsymbol{S}}\boldsymbol{D\omega} + \Pi_{\boldsymbol{S}}^{\perp}\boldsymbol{D\omega}$, then

$$\boldsymbol{z} = \boldsymbol{S}\left(\boldsymbol{a} + \left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\boldsymbol{S}^H\boldsymbol{D\omega a}_i\right) + \Pi_{\boldsymbol{S}}^{\perp}\boldsymbol{D\omega} + \boldsymbol{\epsilon} \tag{30}$$

The first two terms in (30) are orthogonal to each other, hence the CRB for ω equals to the covariance matrix of the LS estimate of $\boldsymbol{\omega}$ in the following decoupled problem

$$\tilde{\boldsymbol{z}} = \Pi_{\boldsymbol{S}}^{\perp}\boldsymbol{D\omega} + \boldsymbol{\epsilon} \tag{31}$$

3. The LS estimate of $\boldsymbol{\omega}$

$$\hat{\boldsymbol{\omega}} = \left[\left(\left(\Pi_{\boldsymbol{S}}^{\perp}\boldsymbol{D}\right)^H\Pi_{\boldsymbol{S}}^{\perp}\boldsymbol{D}\right)_r\right]^{-1}\left[\left(\Pi_{\boldsymbol{S}}^{\perp}\boldsymbol{D}\right)^H\tilde{\boldsymbol{z}}\right]_r \tag{32}$$

4. The CRB for ω is the covariance of $\hat{\boldsymbol{\omega}}$, then

$$\mathrm{CRB}_{\boldsymbol{\omega}} = \frac{\sigma^2}{2}\left[\left(\boldsymbol{D}^H\Pi_{\boldsymbol{S}}^{\perp}\boldsymbol{D}\right)_r\right]^{-1} \tag{33}$$

D. Method 3

The idea of this method is reparameterizing the original model in order to the CRB matrix for a new parameterization becomes block diagonal. Consider the new parameter vector

$$\boldsymbol{\theta}_{\text{new}} = \left[(\boldsymbol{a}_r + \boldsymbol{B}_r\boldsymbol{\omega})^T \quad (\boldsymbol{a}_i + \boldsymbol{B}_i\boldsymbol{\omega})^T \quad \boldsymbol{\omega}^T\right] \tag{34}$$

where $\boldsymbol{B} \triangleq \left(\boldsymbol{S}^H\boldsymbol{S}\right)^{-1}\boldsymbol{S}^H\boldsymbol{D}$. Note that the principal parameters $\boldsymbol{\omega}$ do not change in the new parameter vector. Therefore, the CRB for $\boldsymbol{\omega}$ is still the lower right block of the new CRB for $\boldsymbol{\theta}_{\text{new}}$.

The steps for derivation of the CRB for $\boldsymbol{\omega}$ as follows:

1. From (34), we have

$$\boldsymbol{\theta}_{\text{new}} = \boldsymbol{F\theta} \tag{35}$$

where

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{B}_r \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{B}_i \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{36}$$

2. Using the vector parameter CRB for transformation, we have[1]

$$\mathrm{CRB}_{\boldsymbol{\theta}_{\text{new}}} = \frac{\sigma^2}{2}\boldsymbol{F}\left[\left(\boldsymbol{G}^H\boldsymbol{G}\right)_r\right]^{-1}\boldsymbol{F}^T = \frac{\sigma^2}{2}\left[\left(\left(\boldsymbol{GF}^{-1}\right)^H\left(\boldsymbol{GF}^{-1}\right)\right)_r\right]^{-1} \tag{37}$$

[1]Assume that it is desired to estimate $\boldsymbol{\alpha} = \boldsymbol{g}(\boldsymbol{\theta})$, then we have

$$\mathrm{CRB}_{\boldsymbol{\alpha}} = \frac{\partial \boldsymbol{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\mathrm{CRB}_{\boldsymbol{\theta}}\frac{\partial \boldsymbol{g}(\boldsymbol{\theta})^T}{\partial \boldsymbol{\theta}}$$

3. Substituting

$$\boldsymbol{F}^{-1}=\begin{bmatrix}\boldsymbol{I} & 0 & -\boldsymbol{B}_r\\ 0 & \boldsymbol{I} & -\boldsymbol{B}_i\\ 0 & 0 & \boldsymbol{I}\end{bmatrix} \tag{38}$$

into (37), we obtain $\mathrm{CRB}_{\boldsymbol{\theta}_{\mathrm{new}}}$ in a block diagonal form with the lower right block is $\frac{\sigma^2}{2}\left[\left(\boldsymbol{D}^H\Pi_{\boldsymbol{S}}^{\perp}\boldsymbol{D}\right)_r\right]^{-1}$

IV. Concentrated CRBs

This section considers the CRBs for principal parameters of general systems with unbiased estimator.

A. Problem Statement

Let $\boldsymbol{\theta}\triangleq[\alpha_1,...,\alpha_p,\beta_1,...,\beta_q]^T$ be the $(p+q)\times 1$ parameter vector, where

- $\boldsymbol{\alpha}=[\alpha_1,...,\alpha_p]^T\in\mathbb{R}^{p\times 1}$: the vector of interest parameters
- $\boldsymbol{\beta}=[\beta_1,...,\beta_q]^T\in\mathbb{R}^{q\times 1}$: the vector of nuisance parameters

Let $\boldsymbol{Y}_t\in\mathbb{R}^d$ be the stochastic observation, $t=1,...,N$, and $\boldsymbol{Y}(N)\triangleq\left[\boldsymbol{Y}_1^T,...,\boldsymbol{Y}_N^T\right]^T$.

$\Rightarrow$ Find the CRB for the principal vector $\boldsymbol{\alpha}$ from the observation $\boldsymbol{Y}(N)$.

B. Main Result

Notation:

- $L_t(\boldsymbol{\theta})=L_t(\boldsymbol{\alpha},\boldsymbol{\beta})\triangleq-\log \boldsymbol{f}_{\boldsymbol{Y}_t|\boldsymbol{\alpha},\boldsymbol{\beta}}$ is the log-likelihood function.
- $\hat{\boldsymbol{\beta}}$ is the maximum likelihood estimate of $\boldsymbol{\beta}$ given $\boldsymbol{Y}(N)$ and $\boldsymbol{\alpha}$.
- $\mathcal{L}_t(\boldsymbol{\alpha})\triangleq L_t\left(\boldsymbol{\alpha},\hat{\boldsymbol{\beta}}\right)$ is the concentrated log-likelihood function.
- $\overline{\mathcal{L}}_N(\boldsymbol{\alpha})\triangleq\frac{1}{N}\sum_{t=1}^{N}\mathcal{L}_t(\boldsymbol{\alpha})$ is the average concentrated log-likelihood function.

Under some assumption (see [3]), the CRB for $\boldsymbol{\alpha}$,$\mathrm{CRB}_{\boldsymbol{\alpha}}$, can be found as

$$\lim_{N\to\infty}\frac{\partial^2\overline{\mathcal{L}}_N(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^T}=[\mathrm{CRB}_{\boldsymbol{\alpha}}]^{-1}\quad\text{a.s.} \tag{39}$$

About result gives us the way to find the CRB for interest parameter $\boldsymbol{\alpha}$, i.e.,

1. Evaluate the maximum likelihood estimate $\hat{\boldsymbol{\beta}}$ of the nuisance parameters $\boldsymbol{\beta}$.
2. Substituting $\hat{\boldsymbol{\beta}}$ into the log-likelihood function to obtain the concentrated log-likelihood function $\mathcal{L}_t(\boldsymbol{\alpha})$, then we obtain the average concentrated log-likelihood function $\overline{\mathcal{L}}_N(\boldsymbol{\alpha})$.
3. Differentiating twice the average concentrated log-likelihood function, this result approaches to the inverse of the CRB for $\boldsymbol{\alpha}$, as N grows large.

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