Lecture Note
Key Concepts in Asymptotic Analysis

Erik G. Larsson
March 22, 2011

This lecture note summarizes some key ideas of asymptotic analysis of estimators and provides a heuristic derivation of the main analysis technique. A more rigorous derivation requires additional technical results and can be found in [1], [2] (with slightly different notation).

We are interested in estimating a parameter θ based on n noisy observations. Let the true value of θ be θ_0. The estimator proceeds by finding

$$\hat{\theta}_n = \arg\min_{\theta} f_n(\theta) \tag{1}$$

where $f_n(\theta)$ is the cost function resulting from the collection of data samples.

Application examples

For example, in array processing (single direction of arrival (DOA) estimation) we may have, with $\bar{y}_k$ being the observations,

$$f_n(\theta) = -\left| \frac{1}{n} \sum_{k=1}^{n} \bar{a}^H(\theta)\, \bar{y}_k \right| \tag{2}$$

where $\bar{a}(\theta)$ contains the sensor response to a unit wave front from the direction θ. In source localization, if θ comprises the coordinates (x, y) of the source node, then

$$f_n(x, y) = \sum_{m=1}^{M} \left(r_m - \sqrt{(x - x_m)^2 + (y - y_m)^2} \right)^2 \tag{3}$$

Here M denotes the number anchors and (x_m, y_m) denotes the coordinates of the mth anchor in two dimensions. $r_m = \frac{1}{n}\sum_{k=1}^{n} r_{m,k}$ denotes the average of n measurements at anchor m. In the following discussion, we assume θ to be a scalar quantity.

Consistency

An estimator is said to be consistent if $\hat{\theta}_n \to \theta_0$ as $n \to \infty$. What conditions guarantee that an estimator is consistent?

Fact: If $f_n(\theta)$ converges uniformly to a limiting cost function $f(\theta)$ as $n \to \infty$, i.e.,

$$\sup_{\theta \in \Omega} |f_n(\theta) - f(\theta)| \to 0, \quad n \to \infty$$

over some compact set Ω, then $\hat{\theta}_n \to \theta_0$ as $n \to \infty$. Further, assume that the limiting function $f(\theta)$ has a global minimum at $\theta = \theta_0$.

The same statement can be shown if convergence is in probability, but for the sake of simplicity consider $f_n(\,.\,)$ being deterministic here.

Proof:

With thanks to T.V.K. Chaitanya for help with the typesetting and preparing the figures and to Dr. Saif Mohammed for proofreading.

1) Let $\epsilon > 0$ be given.
2) Define $\Omega^* \triangleq \{\theta : |\theta - \theta_0| < \epsilon\}$
3) Take δ such that

$$\inf_{\theta \in \Omega \backslash \Omega^*} f(\theta) \geq f(\theta_0) + \delta \tag{4}$$

4) Choose n_0 such that

$$|f_n(\theta) - f(\theta)| < \frac{\delta}{3}, \quad n > n_0, \quad \theta \in \Omega \tag{5}$$

5) Then, for $n > n_0$, $\hat{\theta}_n \in \Omega^*$, see Figure 1.
6) Since ϵ is given and arbitrary, the result follows.

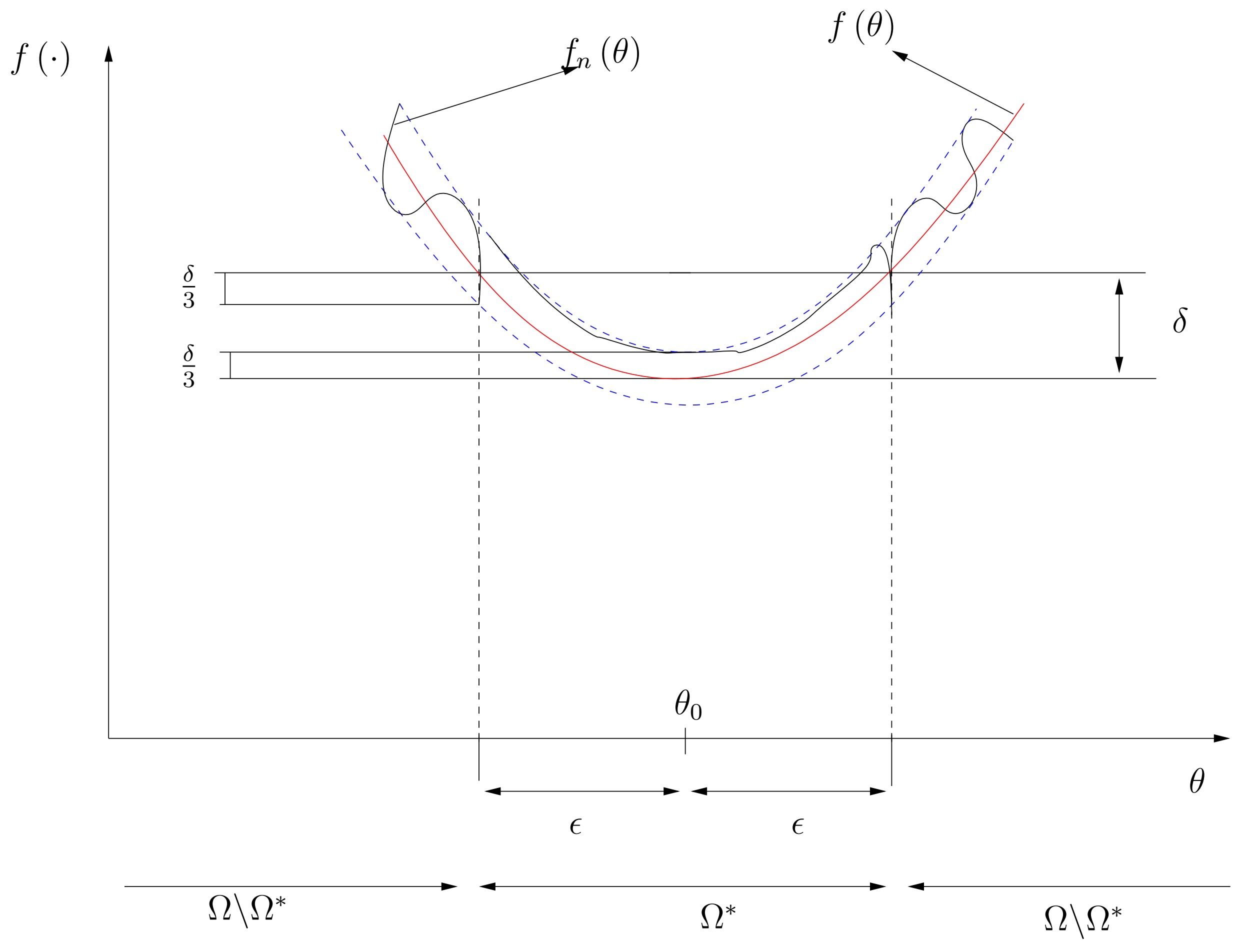

Figure 1. Consistency of $\hat{\theta}_n$ when $f_n(\theta)$ converges uniformly to $f(\theta)$.

Remark: We do need uniform convergence. For example, consider

$$\begin{aligned} f_n(\theta) &= \theta^2 + g_n(\theta), \quad \text{where} \\ g_n(\theta), &= \begin{cases} -1, & \text{for } \theta = (-1)^n + \frac{1}{n} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Here, $\hat{\theta}_n$ does not even tend to a limit, but $f_n(\theta) \to \theta^2 \triangleq f(\theta)$ pointwisely, so $\theta_0 = 0$.

ASYMPTOTIC VARIANCE

Clearly, under appropriate conditions on differentiability, by construction,

$$0 = f_n'\left(\hat{\theta}_n\right)$$

and by a Taylor series expansion around θ_0,

$$0 = f_n'\left(\hat{\theta}_n\right) = f_n'\left(\theta_0\right) + \left(\hat{\theta}_n - \theta_0\right) f_n''\left(\theta^*\right) \tag{6}$$

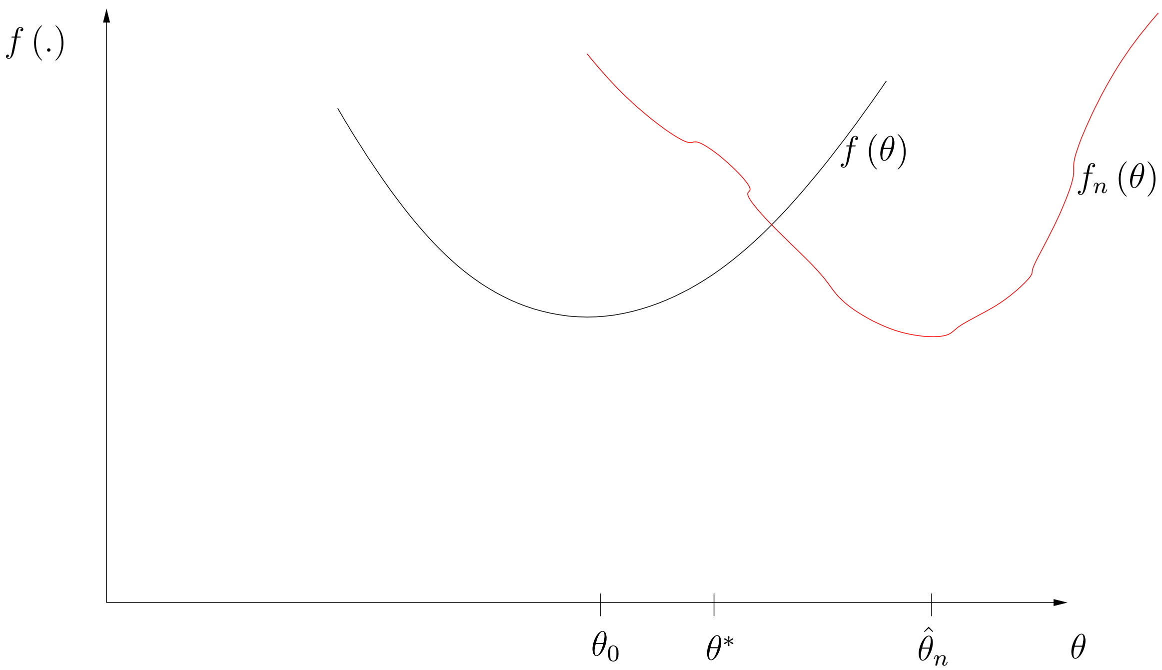

Figure 2. Figure showing $f_n\left(\theta\right)$ and $f\left(\theta\right)$.

where $\left|\theta^* - \theta_0\right| < \left|\hat{\theta}_n - \theta_0\right|$. See Figure 2. Thus, we have

$$\hat{\theta}_n - \theta_0 = -\frac{f_n'\left(\theta_0\right)}{f_n''\left(\theta^*\right)} \tag{7}$$

Here,

- $\theta^* \to \theta_0$ as $n \to \infty$ due to consistency.
- $f_n'\left(\theta_0\right)$ fluctuates around zero.
- $f_n''\left(\theta^*\right)$ fluctuates around $f''\left(\theta_0\right)$ and $f_n''\left(\theta^*\right) \to f''\left(\theta_0\right)$ as $n \to \infty$ w.p. 1

So we can approximate

$$\hat{\theta}_n - \theta_0 \approx -\frac{f_n'\left(\theta_0\right)}{f''\left(\theta_0\right)} \tag{8}$$

Hence

$$\mathbb{E}\left[\left(\hat{\theta}_n - \theta_0\right)^2\right] \approx \frac{\mathbb{E}\left[\left(f_n'\left(\theta_0\right)\right)^2\right]}{\left(f''\left(\theta_0\right)\right)^2} \tag{9}$$

For vector-valued $\boldsymbol{\theta}$, the corresponding result is

$$\text{cov}\left(\hat{\boldsymbol{\theta}}_n\right) = \left(\nabla^2 f\left(\boldsymbol{\theta}_0\right)\right)^{-1} \mathbb{E}\left[\nabla f_n\left(\boldsymbol{\theta}_0\right)\left(\nabla f_n\left(\boldsymbol{\theta}_0\right)\right)^H\right]\left(\nabla^2 f\left(\boldsymbol{\theta}_0\right)\right)^{-1} \tag{10}$$

REFERENCES

[1] L. Ljung, *System identification: Theory for the user*, Prentice–Hall, 1999.
[2] T. Söderström and P. Stoica, *System Identification*, available online at http://user.it.uu.se/~ts/sysidbook.pdf.